A NONLINEAR DIFFERENTIAL-DIFFERENCE EQUATION OF GROWTH

Dunham Laboratory
Yale University
New Haven, Connecticut

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Preface

This report is the fifth concerned with research accomplished in connection with Navy Contract Nonr-433(00), between Dunham Laboratory, Yale University, and the Office of Naval Research, Department of the Navy. In this report is given a discussion of the solution for a nonlinear differential-difference equation. Methods of attacking equations of this general sort are but imperfectly developed, and only approximate solutions can be obtained. The particular equation considered here may apply to several physical phenomena of interest, and the mathematical analysis is of interest in itself.

The research was carried on by W. J. Cunningham, with the assistance of J. G. Skalnik; the report was written by the undersigned.

W. J. Cunningham

New Haven, April, 1954

Table of Contents

List of Figures

List of Symbols

a parameter in simple growth equation

$A = A(t)$ amplitude of oscillation for oscillatory solution

A_o initial value of A at $t = 0$

A_s steady-state value of A

b parameter in modified growth equation

C arbitrary constant

$g = b/a = 1/\beta$

$h = (x_o^{-1} - g)$

m slope of solution curve on z-x plane

$p = (a\tau^2)^{-1} (a\tau - 1)$

P mean value of solution in Appendix

$q = b^2/4a^3\tau^2$

Q amplitude of oscillatory solution in Appendix

t independent variable in equation, considered as time

t_o time for beginning step-by-step computation

Δt interval for one step of computation

T period of oscillation

u small change in x, measured from x_s

v small change in y, measured from y_s

w variable in linearized Bernoulli equation

$x = x(t)$ dependent variable of equation

$\dot{x} = dx/dt$

$\ddot{x} = d^2x/dt^2$

x_o initial value of x at $t = 0$

x_s value of x at a singular point

$x_1 = x_1(t)$ solution for Eq. (4) with $\tau = 0$

$x_2 = x_2(t)$ approximate solution for Eq. (9), with $x < 0$ and τ small

$y = \dot{x}$

y_s value of y at a singular point

$z = \tau\dot{x}$

z_s value of z at a singular point

$\alpha = 2^{1/2}/\tau$

$\beta = a/b$

θ phase angle of oscillation

λ characteristic root of differential equation

τ parameter in modified growth equation, the delay time

$\phi = (\alpha t + \theta)$

ω angular frequency of oscillatory solution in Appendix

Abstract

Certain physical phenomena appear to be described by the nonlinear differential-difference equation

$$dx(t)/dt = \left[a - b\,x(t - \tau)\right] x(t)$$

where a, b, and τ are positive real constants. This is an equation of growth in which the growth rate of a quantity depends in part upon the value of that quantity at some earlier time. Methods of obtaining exact solutions for this sort of equation are unknown.

Approximate solutions for the equation are obtained by several analytical methods. Variable x can never go through the value zero, and thus reverse its algebraic sign. If x is positive and the product $a\tau$ is small, solutions approach a limiting value a/b, either monotonically or with a decaying oscillation. If product $a\tau$ is larger, a steady-state oscillation of definite amplitude occurs. If x is negative, solutions run off to negative infinity.

Examples of solutions for particular values of the parameters are obtained with an analog computer.

I. Phenomena and Equation under Consideration

There are certain natural phenomena in which the magnitude of some quantity increases at a rate proportional to the magnitude itself. A simple example is the growth in population of an organism,[1] where the number of new individuals appearing within any given short time interval depends upon the number of individuals present at the beginning of the interval. Certain chemical or neuclear reactions[2] may operate in a similar way, with the rate of reaction proportional to the amount of end product that is present.

Phenomena such as these can be described mathematically by the differential equation

$$dx/dt = ax \tag{1}$$

where x is a measure of the product in question, t is time, and a is a positive real constant, the relative growth rate. This equation is sometimes referred to as the equation of growth. Its solution can be written as

$$x = x_0 \exp(at) \tag{2}$$

where x_0 is the value of x existing at zero time. Curves representing x as a function of t have the well-known exponential shape of Fig. 1, where a family of curves is shown for constant x_0 but several values of a. As t increases without limit, so also does x increase without limit, the rate of increase becoming proportionally larger.

There are certain phenomena which appear to be governed initially by an equation such as Eq. (1). However, as the quantity represented by x increases, some effect comes into play which reduces its rate of

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1. H. Margenau and G. M. Murphy. Mathematics of Physics and Chemistry, (Van Nostrand, New York, 1943), p. 33

2. W. Jost, Explosion and Combustion Processes in Gases, (McGraw-Hill, New York, 1946), p. 282

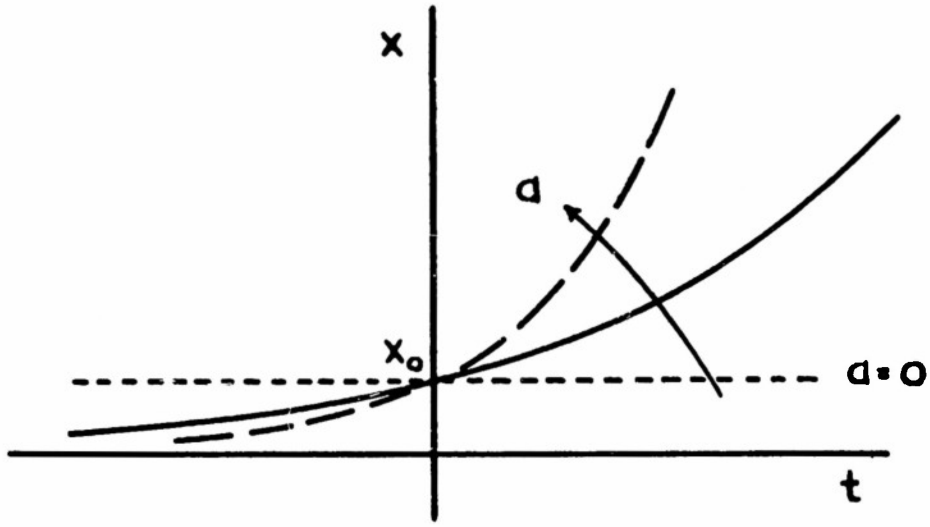

Fig. 1 Exponential functions satisfying Eq. (1).

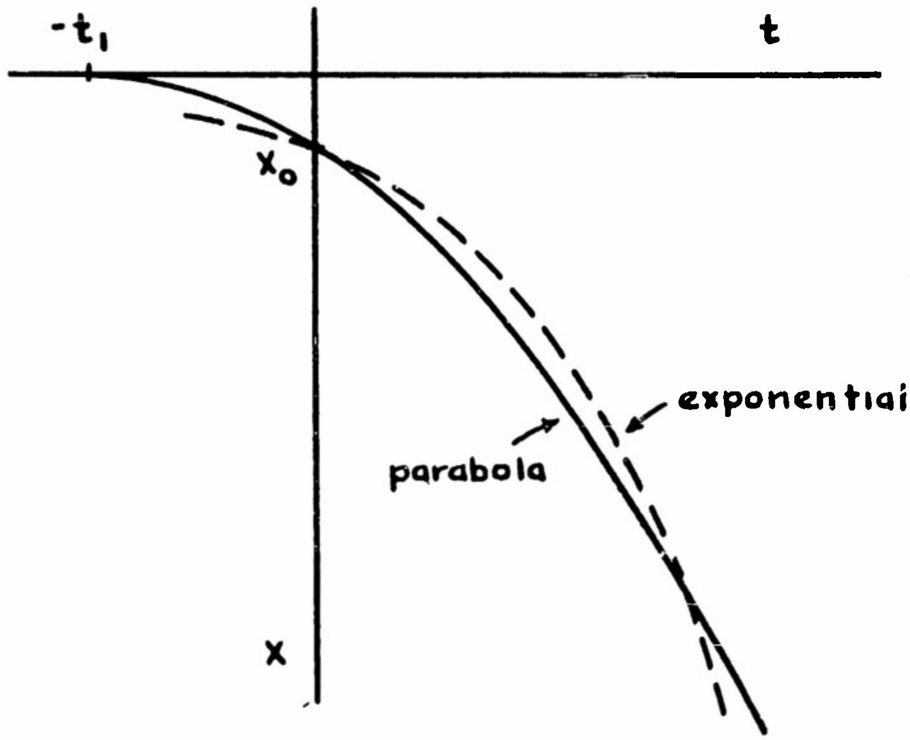

Fig. 2. Exponential curve approximating a parabolic path.

increase. Ultimately x may approach a constant value. This might be the case for a colony of biological organisms in circumstances where the food supply or living space is limited. As the population of the organisms approaches the maximum which can be supported continuously, the rate of growth must be reduced in some way. One hypothesis that has been suggested[3] is that the members of the population somehow recognize that crowding is soon to occur. Upon making this discovery, they try consciously to reduce their reproduction rate. Because of a finite gestation time, the actual birth rate is not lowered until a definite time after an effort has been made to lower it. An equation describing this kind of operation is

$$dx(t)/dt = \left[a - b\,x(t-\tau)\right] x(t) \tag{3}$$

where a and b are positive real constants, τ is a constant delay time, and $x(t)$ and $x(t-\tau)$ are values of x at the instants t and $(t-\tau)$, respectively. The effective growth rate in Eq. (3) is $\left[a - b\,x(t-\tau)\right]$. It is less than parameter a by an amount proportional to the value of x existing at the time $(t-\tau)$, earlier than the time t at which the derivative $dx(t)/dt$ is evaluated.

Another example where Eq. (3) might occur is in the control of some reaction which fundamentally is governed by Eq. (1). In order to prevent the reaction running away, with catastrophic results perhaps, some modification is intentionally introduced into the system to reduce the reaction rate. The controlling mechanism, which senses the rate of reaction and takes steps to change it, requires a finite time to operate. If the delay time is of fixed value, τ, the equation applying to the system is Eq. (3).

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3. This equation appears to have been introduced in the paper, G. E. Hutchinson, "Circular Causal Systems in Ecology", Annals of the New York Academy of Sciences, 50, 221, (1948).

Still another example where this equation might arise is in the determination of the path of a controlled projectile. A mass particle, falling freely in a constant gravitational field with no retarding effects, follows a parabolic path. Furthermore, the curve of height as a function of time also is parabolic, since the horizontal component of velocity remains constant, and the horizontal position is proportional to time. An example of the relation between height and time is shown in Fig. 2. Coordinates x and t are chosen so that the vertex of the parabola is located at $x = 0$ and $t = -t_1$. For convenience, positive x is plotted below the t-axis. At zero time the particle is at the point $x = x_0$ and $t = 0$. The parabolic curve can be approximated over a part of its length by an exponential curve suitably chosen. Such a curve is shown also in Fig. 2. An actual mass particle, falling in the gravitational field of the earth, is retarded by the effect of resistance with the atmosphere. This retardation ultimately makes the vertical component of velocity constant, so that the curve of Fig. 2 would approach asymptotically a straight line of constant slope. The curve then could not be approximated by an exponential curve.

Instead of the particle falling freely, it might be subject to control, the intent of which is to make the path become horizontal at some definite value of x. Again the control system requires a finite time to operate. If this delay time is constant, and if the free fall of the projectile is assumed to be essentially exponential in shape, Eq. (3) may describe the path which the projectile follows.

Equations similar to Eq. (3) occur in economic studies[4] of business cycles, where time delays occur in various steps of the business operations.

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4. R. M. Goodwin, Econometrica, 19, 1, (1951)

In general, any phenomenon in which some quantity tends to grow at an ever-increasing rate, but is subjected to a throttling effect involving a constant time delay, may be described by Eq. (3).

The analysis of nonlinear differential-difference equations[5] seems not to have been studied in great detail, so that methods of finding exact solutions for Eq. (3) are unknown. Qualitatively, however, it is apparent that if $x(t)$ ever vanishes, the value of $dx(t)/dt$ also must vanish. If this occurs, x can never become different from zero again. Thus, if $x(t)$ is not zero, it must always retain the same algebraic sign, and the signs of both $x(t)$ and $x(t - \tau)$ must be identical.

If $x(t)$ is positive, the sign of $dx(t)/dt$ may be either positive or negative, depending upon the relative magnitudes of the terms on the right side of Eq. (3). Thus, the appearance of oscillations with $x(t)$ positive is allowed. A steady value, $x(t) = x(t - \tau) = a/b$, for which $dx(t)/dt = 0$, also is a possible solution. If $x(t)$ is negative, the sign of $dx(t)/dt$ always is negative, and the solution can only go to negative infinity.

II. Degenerate Case for $\tau = 0$

A simple case of Eq. (3), and a profitable one to take as a starting point, is that for which the delay time τ is zero, so that the equation is[6]

$$dx/dt = ax - bx^2. \qquad (4)$$

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5. R. Bellman and J. M. Danskin, "Stability Theory of Differential-Difference Equations," Proceedings of Symposium on Nonlinear Circuit Analysis, (Polytechnic Institute of Brooklyn, New York, 1953). p. 107. This reference contains a long bibliography.

6. This is the Verhulst-Pearl equation. See A. J. Lotka, Elements of Physical Biology, (Williams and Wilkins, Baltimore, 1925), p. 64.

This equation can be studied initially by considering a plot of the new variable $y \equiv dx/dt = \dot{x}$ as a function of x. Such a plot is the Poincare' phase plane,[7] widely used in studying second-order equations. The phase-plane curve for Eq. (4) is shown in Fig. 3.

There are two points, $x = 0$ and $x = a/b$, where $y = 0$. These are points of equilibrium, since no change can occur if x initially has either of these values. If x initially has some other value, changes will occur. If y is positive, x changes to become more positive, and vice versa. Thus, changes in x with respect to time take place in the directions indicated in Fig. 3. The point $x = 0$ is unstable in the sense that x tends to run away from this point with time. The point $x = a/b$ is stable, since x tends to converge toward this point. Both points are termed nodal points, with the curve of y as a function of x approaching from a definite direction.

An exact solution for Eq. (4) can be found, considering it as an example of Bernoulli's equation. The substitution $w = 1/x$ is made, giving the linear equation

$$dw/dt + aw = b \tag{5}$$

which has the solution

$$w = b/a + C \exp(-at) \tag{6}$$

with C an arbitrary constant. Solution of Eq. (4) is, then,

$$x = \left[b/a + (1/x_o - b/a) \exp(-at)\right]^{-1} \tag{7}$$

where $x = x_o$ at $t = 0$.

The nature of this solution can readily be compared with Fig. 3. As t becomes infinitely large, x always approaches the value $x = a/b$ monotonically. If x_o is positive, the curve for x as a

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7. N. Minorsky, Nonlinear Mechanics, (J. W. Edwards, Ann Arbor, 1947) Part I

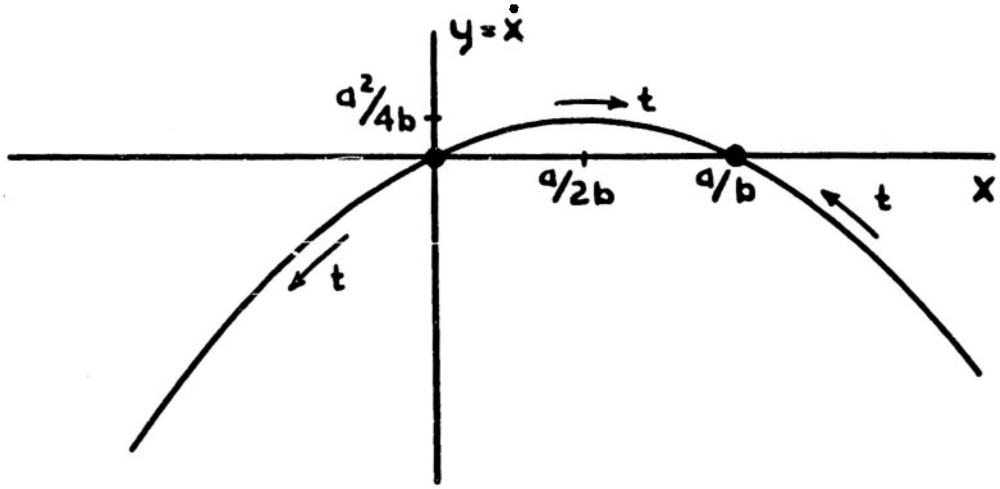

Fig. 3 Phase plane for Eq. (4).

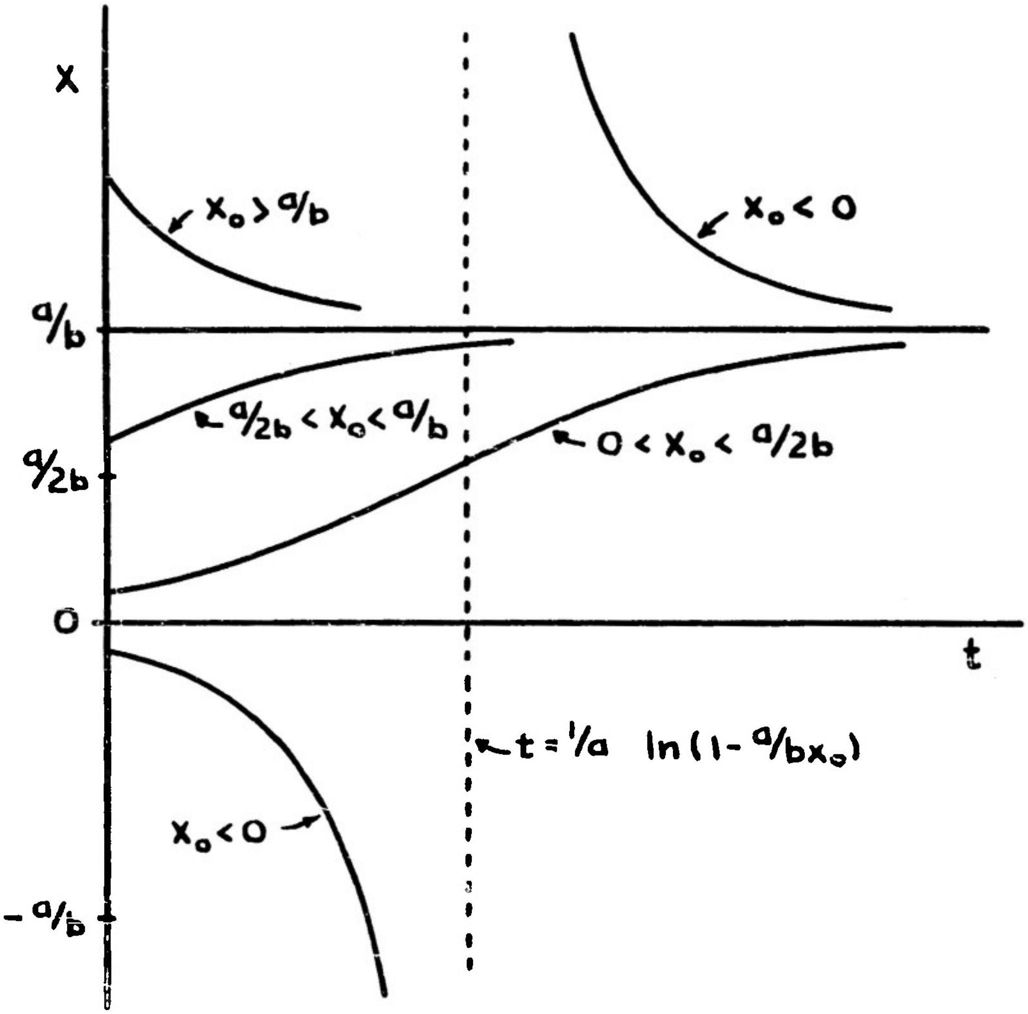

Fig. 4 Solutions for Eq. (4).

function of t is continuous. Its curvature is zero at the value of t making $d^2x/dt^2 = 0$, which is the same as $dy/dx = 0$. This value is $t = a^{-1} \ln(a/bx_o - 1)$ at which instant $x = a/2b$, and the slope of the curve has its maximum positive value at this point. If x_o is negative, the curve for x goes to infinity for the value of t causing the quantity in the bracket of Eq. (7) to vanish. This value is $t = a^{-1} \ln(1 - a/bx_o)$.

A family of solution curves for Eq. (4) is shown in Fig. 4. For small values of x, the rate of growth is determined primarily by parameter a; for large values of x it is determined primarily by parameter b. The ultimate value of x depends upon the ratio a/b. These solution curves apply to Eq. (3) with $\tau = 0$, and with both a and b positive. Extension to the case of reversed sign for these parameters is self-evident.

III. Differential Equation Approximately Equivalent

III. 1. Derivation from differential-difference equation

It is difficult to study the differential-difference equation, Eq. (3), because of the term $x(t - \tau)$ which is evaluated at a time different from the other terms, $x(t)$ and $dx(t)/dt$. A differential equation, with all terms evaluated at the same instant, is more easily analyzed. The term $x(t - \tau)$ can be expressed by the Taylor's series

$$x(t - \tau) = x(t) - \tau dx(t)/dt + (\tau^2/2)d^2x(t)/dt^2 - (\tau^3/6)d^3x(t)/dt^3 + \cdot\cdot\cdot\cdot\cdot \tag{8}$$

where each term on the right side is evaluated at the same instant, t. If the delay time, τ, is sufficiently small, its higher powers become still smaller and the series of Eq. (8) may converge fairly rapidly. Then a good approximation to the value of $x(t - \tau)$ may be had from

only the first few terms of the series. If either τ or the higher-order derivatives are not sufficiently small, the series converges slowly, and many terms must be retained for accuracy.

If only the first three terms of Eq. (8) are substituted into Eq. (3), the result is

$$\ddot{x} - \alpha^2\tau\dot{x} + \alpha^2\dot{x}/bx + \alpha^2 x = \alpha^2\beta \qquad (9)$$

where $\alpha^2 \equiv 2/\tau^2$, $\beta \equiv a/b$, and all terms are evaluated at the same instant. If $\tau = 0$, Eq. (9) reduces to Eq. (4) as it should.

Equation (9) is a second-order, nonlinear differential equation, representing a system with a single degree of freedom. With certain choices of parameters its solutions are oscillatory, but only a single mode of oscillation at a single frequency can occur at a given time. A differential equation of infinite order would have resulted if all terms of Eq. (8) had been used. Such an equation would represent a system with an infinite number of degrees of freedom, having the possibility of simultaneous oscillation at an infinity of frequencies. The modification of the differential-difference form, Eq. (3), to the pure differential form, Eq. (9), has brought a simplification, but at the expense of losing the possibility of simultaneous oscillation at several frequencies. Techniques for studying a second-order equation, such as Eq. (9), are well developed; equations of higher order are more difficult to analyze. It is not unreasonable to hope that solutions for Eq. (9) will be similar to those of Eq. (3), although exact correspondence cannot be expected.

Equation (9) itself can be interpreted as applying to a physical system somewhat different from the examples used in connection with Eq. (3). If the nonlinear term, $\alpha^2\dot{x}/bx$, of Eq. (9) is neglected, the

remaining terms are linear, and the equation has the solution

$$x = \beta + C \exp(t/\tau) \cos(t/\tau + \theta) \tag{10}$$

where C and θ are arbitrary constants. This solution is the sum of a constant term and an oscillation with amplitude growing with time. The nonlinear term of Eq. (9) represents damping which varies inversely with the value of x. Thus the solution of nonlinear Eq. (9) might be expected to resemble Eq. (10), but with large damping occurring at those instants when x is small. It is not unlikely that the combination of this damping with the growing exponential factor of Eq. (10) will result in the appearance of a limit cycle,[8] an oscillation of fixed amplitude determined solely by the parameters of the equation and independent of initial conditions. Thus, Eq. (9) might apply to an oscillator with a linear negative-damping term, and a positive-damping term varying inversely with the value of x. Since the positive damping would become infinite if x ever goes to zero, it is evident that a steady oscillation can occur only about some non-zero mean value of x, and this is provided by the constant term of Eq. (9).

Equation (9) may be compared with the well-known van der Pol equation,[9] $\ddot{x} - \varepsilon(1 - x^2)\dot{x} + x = 0$, which also has solutions in the form of limit cycles. In the van der Pol equation, positive damping for large amplitudes of oscillation occurs symmetrically at either extreme of the cycle. In Eq. (9), positive damping occurs asymmetrically, being large only for instantaneous values of x on the negative side of the mean value, β.

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8. N. Minorsky, ref. 7, p. 62

9. N. Minorsky, ref. 7, p. 113

III. 2. Singularity at $x_s = \beta$, $y_s = 0$.

The qualitative nature of the solutions for Eq. (9) can be determined by studying the singular points[10] of the first-order equation resulting from the substitution, $y = \dot{x}$, so that $\ddot{x} = \dot{y} = y\, dy/dx$. This equation is

$$\frac{dy}{dx} = \frac{\alpha^2[(\tau - 1/bx)y - x + \beta]}{y} \quad . \tag{11}$$

Its solution can be represented as a curve on the phase plane having as axes the coordinates x and y. Singular points are located at those values of x and y which make both numerator and denominator of Eq. (11) vanish simultaneously. Only one singular point is so determined; its coordinates are $x_s = \beta$ and $y_s = 0$.* The nature of the solutions near this singularity can be found by replacing x and y with

$$x \longrightarrow x_s + u = \beta + u$$

$$y \longrightarrow y_s + v = v$$

where u and v are small increments. If these substitutions are made in Eq. (11), and only linear terms are retained, the result is

$$\frac{dv}{du} = \frac{\alpha^2[(\tau - 1/a)v - u]}{v} \quad . \tag{12}$$

Since $dv/du = (dv/dt)/(du/dt) = (1/v)(d^2u/dt^2)$, Eq. (12) is equivalent to the equation

$$d^2u/dt^2 - \alpha^2(\tau - 1/a)\, du/dt + \alpha^2 u = 0 \, . \tag{13}$$

Solutions for this linear second-order equation depend upon the roots, λ_1 and λ_2, for the characteristic equation

$$s^2 - \alpha^2(\tau - 1/a)\, s + \alpha^2 = 0 \, . \tag{14}$$

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10. N. Minorsky, ref. 7, Ch. III

* It is worth remarking that a second singularity is located at the origin, but that this point is not a simple singularity and cannot be discussed in the same manner as the first singular point. It is considered briefly in Sec. III. 5.

These roots are

$$\lambda_1, \lambda_2 = (a\tau^2)^{-1} \left\{(a\tau - 1) \pm \left[(a\tau - 1)^2 - 2a^2\tau^2\right]^{1/2}\right\}. \qquad (15)$$

Solutions are non-oscillatory if the roots are real quantities, that is, if $\left[(a\tau - 1)^2 - 2a^2\tau^2\right] > 0$. This will be the case if $(-2^{1/2} - 1) < (a\tau) < (2^{1/2} - 1)$; otherwise oscillations will occur. Solutions are stable, in the sense that they do not increase without bound as time increases, if $(a\tau^2)^{-1} (a\tau - 1) < 0$. Actually, of course, parameters a and τ were assumed positive at the beginning of the discussion, but they might be allowed to become negative under some conditions. The qualitative nature of solutions is depicted graphically in Fig. 5. The notation used here is that conventional in referring to the nature of a singularity. A focus refers to an oscillating solution with amplitude either decreasing or increasing; a node refers to a solution approaching a limiting value monotonically. The solution found in Sec. II, for $a > 0$ and $\tau = 0$, approaches the singular point monotonically, and the point is a stable node.

According to the theory of Liapounoff,[11] the nature of solutions for the nonlinear equation, Eq. (9), are similar to those of the linear equation, Eq. (13), so long as the variables u and v are sufficiently small. Thus, in the nieghborhood of the singular point, solutions of Eq. (9) have the properties illustrated in Fig. 5.

III. 3. Approximate solution by variation of parameters, about (x_s, y_s)

Still more information about the solutions of Eq. (9) can be found by applying the method of variation of parameters.[12] This method is useful where an oscillating solution occurs with only small changes in either amplitude or phase taking place within a cycle. In applying

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11. N. Minorsky, ref. 7, p. 51
12. N. Minorsky, ref. 7, Ch. X

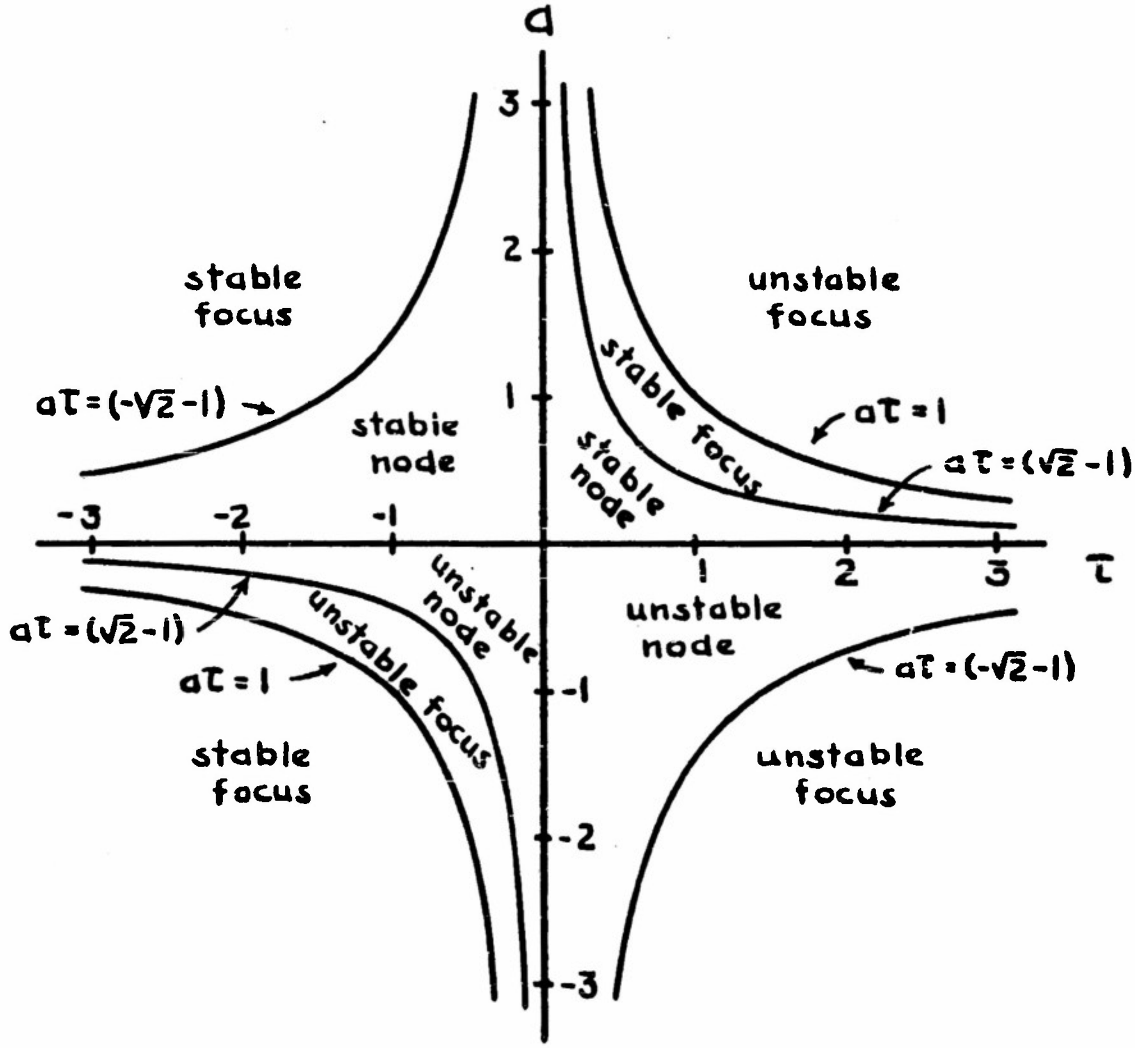

Fig. 5 Stability diagram for solutions of Eq. (9) near the point $X_s = \beta$, $y_s = 0$.

the method, it is convenient to rewrite Eq. (9) as a pair of first-order equations,

$$\dot{x} = y$$
$$\dot{y} = \alpha^2(\tau - 1/bx)y - \alpha^2 x + \alpha^2\beta \quad (16)$$

If the term in y, with its nonlinearity, is omitted from the second of these equations, a generating solution is found as

$$x = \beta + A\sin(\alpha t + \theta) = \beta + A\sin\phi$$
$$y = \alpha A\cos\phi \quad (17)$$

where A is the amplitude and θ is the phase angle found from initial conditions, and $\phi \equiv (\alpha t + \theta)$. This generating solution is then used in the complete form of Eq. (16), allowing both A and θ to become time-dependent instead of being mere constants. The result of this substitution is

$$\dot{A}\sin\phi + A\dot{\theta}\cos\phi = 0$$
$$\dot{A}\alpha\cos\phi - A\alpha\dot{\theta}\sin\phi = \alpha^3\tau A\cos\phi - \frac{\alpha^3 A\cos\phi}{b\beta[1 + (A/\beta)\sin\phi]} \quad (18)$$

Solution for $\dot{A}$ gives

$$\dot{A} = \alpha^2 A\cos^2\phi\left[\tau - a^{-1}(1 + A/\beta\sin\phi)^{-1}\right].$$

If $A/\beta \ll 1$, approximately

$$(1 + A/\beta\sin\phi)^{-1} = 1 - A/\beta\sin\phi + A^2/\beta^2\sin^2\phi + \cdots \quad (19)$$

and $\dot{A}$ becomes

$$\dot{A} = \alpha^2\tau A\cos^2\phi - \alpha^2 A/a\cos^2\phi + \alpha^2A^2/a\beta\cos^2\phi\sin\phi$$
$$- \alpha^2A^3/a\beta^2\cos^2\phi\sin^2\phi + \cdots .$$

If the amplitude changes but slowly, only the average rate of change over a cycle need be considered. Since

$$\cos^2\phi = \tfrac{1}{2}(1 + \cos 2\phi)$$
$$\cos^2\phi\sin\phi = \tfrac{1}{4}(\sin\phi + \sin 3\phi)$$
$$\cos^2\phi\sin^2\phi = \tfrac{1}{8}(1 - \cos 4\phi)$$

the average value of $\dot{A}$ over a cycle is

$$(\dot{A})_{av} = (a\tau^2)^{-1} (a\tau - 1) A - b^2A^3/4a^3\tau^2$$
$$= pA - qA^3 \tag{20}$$

where $p \equiv (a\tau^2)^{-1} (a\tau - 1)$ and $q \equiv b^2/4a^3\tau^2$

In a similar manner, $\dot{\theta}$ can be determined, and its average found over a cycle of ϕ. The result is

$$(\dot{\theta})_{av} = 0. \tag{21}$$

If the average value of $\dot{A}$, given in Eq. (20), is used, the variation of amplitude with time can be found. This equation, another example of Bernoulli's equation, can be made linear by the substitution $w = A^{-2}$. The resulting linear equation is

$$dw/dt + 2pw = -2q. \tag{22}$$

Its solution is

$$w = C \exp(-2pt) + q/p \tag{23}$$

and thus,

$$A = \left[C \exp(-2pt) + q/p\right]^{-(1/2)} \tag{24}$$

where C is an arbitrary constant. If $A = A_o$ at $t = 0$, amplitude A is given as a function of time by

$$A = A(t) = \left[q/p + (A_o^{-2} - q/p) \exp(-2pt)\right]^{-(1/2)}. \tag{25}$$

The solution just obtained is only approximate because just the first three terms of the infinite series of Eq. (19) were used. This approximation is good if $A/\beta \ll 1$, which is true only near the beginning of the growth of A and provided $A_o \ll \beta$. As the instantaneous amplitude increases, the solution is increasingly in error.

Quantity p in Eq. (25) may be either positive or negative, as product $a\tau$ is greater or less than unity, respectively. If p is positive, amplitude A approaches a steady-state value,

$$A_s = (p/q)^{1/2} = 2\beta(a\tau - 1)^{1/2} \tag{26}$$

as t becomes infinite. If initially A_o is small, the amplitude grows, with its maximum rate of change occurring when $\ddot{A} = 0$, at which instant $A = A_s/3^{1/2}$. If p is negative, amplitude A vanishes as t becomes infinite.

Plots of the variation of $\dot{A}$ with A, and of A with t, as given by Eqs. (20) and (25), are shown in Fig. 6. Since Eq. (21) shows that the average frequency is constant, the approximate solution for Eq. (9), just found, is

$$x = \beta + A(t) \sin(2^{1/2}t/\tau + \theta_o) \tag{27}$$

where $A(t)$ is given by Eq. (25) and plotted in Fig. 6, and θ_o is a phase angle determined by initial conditions.

Limitations can be set on the value of product $a\tau$ allowed for this solution. The assumption is made initially that the change in amplitude per cycle of the oscillation must be small. If p is positive, the maximum value $\dot{A}_m$ for the average rate of change of amplitude occurs for the amplitude $A_m = (p/3q)^{1/2}$. The relative rate of growth is then, from Eq. (20), $\dot{A}_m/A_m = 2p/3$. The period of the oscillation is $T = 2\pi/\alpha = 2^{1/2}\pi\tau$. Thus, the maximum relative change in amplitude per cycle is nearly

$$T\,\dot{A}_m/A_m = (2^{1/2}\pi\tau)(2p/3) = (2^{3/2}\pi/3)(a\tau - 1)(a\tau)^{-1}. \tag{28}$$

This quantity must be less than unity, say, to meet the assumption inherent in the method of solution. Thus, the requirement is that

$$a\tau < \left[1 - (2^{3/2}\pi/3)^{-1}\right]^{-1} \doteq 3/2. \tag{29}$$

If p is positive, the amplitude approaches the steady value, $A_s = 2\beta(a\tau - 1)^{1/2}$. This amplitude exceeds the value β for $a\tau > 5/4$, and would then require instantaneous values of x to change algebraic sign. Since sign changes cannot occur because of infinite damping at $x = 0$, a more realistic upper limit for $a\tau$ in the solution is $a\tau < 5/4$.

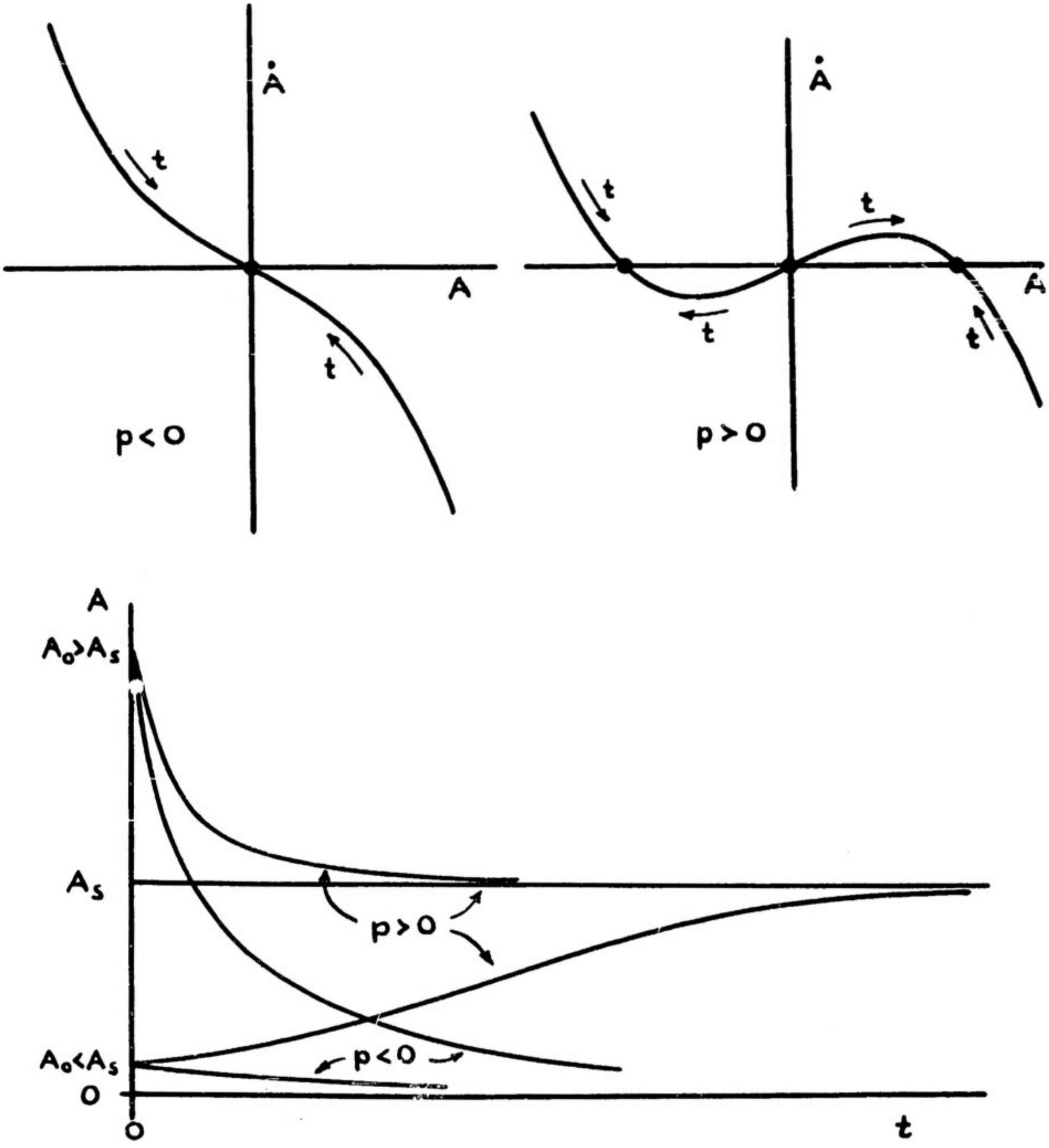

Fig. 6 Solutions for amplitude of oscillation in Eq. (27).

If p is negative, the amplitude decays with time. According to Eq. (20), the maximum rate of decay occurs for the largest, or initial, amplitude. A rather crude estimate of the lower limit for $a\tau$ can be obtained by disregarding the nonlinear term in Eq. (20), in which case the relative rate of decay is $\dot{A}/A = p$, independent of the amplitude. The relative change in amplitude per cycle is, then,

$$T\,\dot{A}/A = (2^{1/2}\pi\tau)(a\tau - 1)(a\tau^2)^{-1} . \tag{30}$$

This quantity must be greater than minus one, say, so that

$$a\tau > \left[1 + (2^{1/2}\pi)^{-1}\right]^{-1} \doteq 4/5. \tag{31}$$

Thus, in order that the approximate solution found by variation of parameters be reasonably accurate, the value of product $a\tau$ must fall within the limits

$$4/5 < (a\tau) < 5/4.$$

For these conditions, the approximate solution is an oscillation, approximately sinusoidal with angular frequency $2^{1/2}/\tau$, having its mean value at $x = \beta$. Its amplitude decays if $a\tau < 1$. The amplitude approaches a steady value, $A_s = 2\beta(a\tau - 1)^{1/2}$, if $a\tau > 1$.

III. 4. Approximate solution by iteration, about (x_s, y_s)

The approximate solution obtained by the method of variation of parameters consists of an oscillation, essentially sinusoidal in waveform, with its amplitude varying in time. An oscillatory steady state is achieved if $a\tau > 1$. More information about the waveform in the steady state can be found by a process of iteration.[13]

If the terms in $\dot{x}$ are omitted from Eq. (9), it becomes

$$\ddot{x} + \alpha^2 x = \alpha^2 \beta \tag{32}$$

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13. J. J. Stoker, Nonlinear Vibrations, (Interscience Publishers, New York, 1950), p. 83

which has the solution

$$x = \beta + A \sin \alpha t \tag{33}$$

subject to the initial conditions of $x = \beta$, $\dot{x} = \alpha A$ at $t = 0$. This generating solution can now be put into those terms of Eq. (9) in which $\dot{x}$ appears so as to give

$$\ddot{x} + \alpha^2 x = \alpha^2\beta + \alpha^3\tau A \cos \alpha t$$
$$- (\alpha^3 A/b\beta) \cos \alpha t \,(1 - A/\beta \sin \alpha t$$
$$+ A^2/\beta^2 \sin^2 \alpha t + \cdot \cdot \cdot \cdot \cdot) \tag{34}$$

where the series is accurate only if $A/\beta \ll 1$. Use can be made of the identities

$$\sin \alpha t \cos \alpha t = (\tfrac{1}{2}) \sin 2\alpha t$$

$$\sin^2 \alpha t \cos \alpha t = (\tfrac{1}{4})(\cos \alpha t - \cos 3\alpha t)$$

to give

$$\ddot{x} + \alpha^2 x = \alpha^2\beta + \alpha^3 A(\tau - 1/b\beta - A^2/4b\beta^3) \cos \alpha t$$
$$+ (\alpha^3 A^2/2b\,\beta^2) \sin 2\alpha t + \cdot \cdot \cdot \cdot \tag{35}$$

In order to avoid a secular term in the solution, something which cannot occur since it grows indefinitely with time, the coefficient of $\cos \alpha t$ must be zero, or

$$A = 2\beta(a\tau - 1)^{1/2}. \tag{36}$$

This is the steady-state amplitude of oscillation, and is the same result found in Eq. (26).

Solution for Eq. (35) is, then

$$x = \beta + A \sin \alpha t - \alpha A^2/6b\beta^2 \sin 2\alpha t + \cdot \cdot \cdot \cdot \tag{37}$$

where the coefficient of a possible term in $\cos \alpha t$ is made zero to keep $x = \beta$ at $t = 0$.

The waveform of the steady-state oscillation is approximated by Eq. (37), and considerable second-harmonic distortion is seen to be

present. The relative amplitude of the second harmonic compared with the fundamental component is

$$2\text{nd}/1\text{st} = \alpha A/6b\beta^2 = (2^{1/2}/3)(a\tau - 1)^{1/2}(a\tau)^{-1}. \tag{38}$$

The largest value of $a\tau$ which will keep instantaneous x positive (as is necessary) is $a\tau = 5/4$. For this value, the relative second harmonic is $2\text{nd}/1\text{st} \doteq 1/5$.

A sketch of the resulting waveform is shown in Fig. 7. The presence of the harmonic actually makes x go negative momentarily in this example. The rise in x from values just greater than zero to large values occurs more slowly than does the drop from large values to small ones.

It should be recognized, of course, that Eq. (37) is only approximate because of the gross assumptions made in its derivation. It applies reasonably well only between the limits $1 < a\tau < 5/4$.

III. 5. Solution by isocline construction

It is profitable to study in still more detail the solutions of Eq. (9) as represented graphically on the phase plane. In carrying the analysis further it is convenient to normalize the quantity $y = \dot{x}$ by defining a new variable $z \equiv \tau y = \tau\dot{x}$. The dimensions of z are then the same as those of x. In terms of z, Eq. (11) becomes

$$\frac{dz}{dx} = \frac{2[(1 - 1/b\tau x)z - x + \beta]}{z} \tag{39}$$

This equation gives the slope of a solution curve at any point in the z-x plane. If this slope is assigned a constant value, say m, the locus of those combinations of z and x giving this assigned slope can be found. This locus is the isocline[14] curve connecting points

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14. N. Minorsky, ref. 7, p. 20

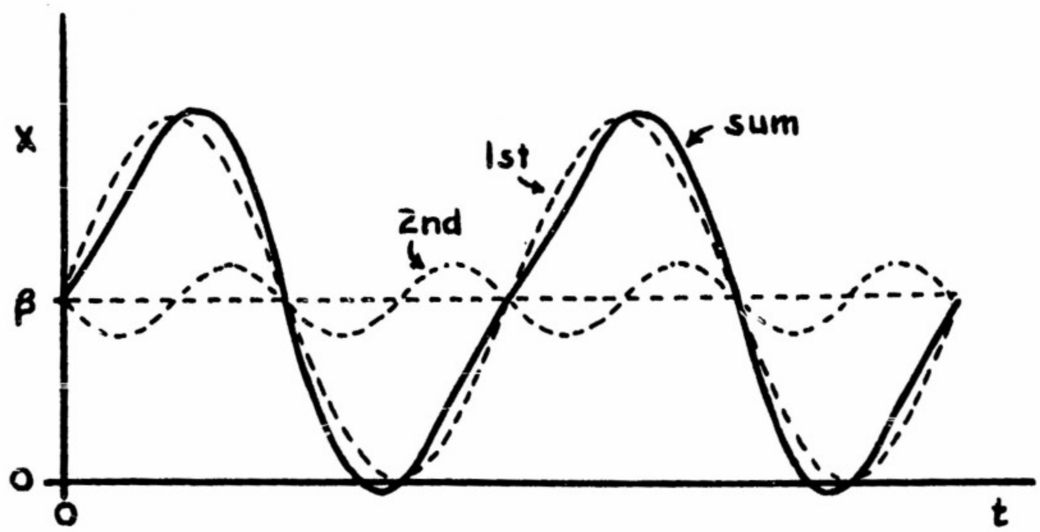


Fig. 7 Approximate steady-state solution for Eq. (9), with $a\tau = 5/4$.

of constant slope, m. Its equation is

$$z = \frac{b\tau x(\beta - x)}{1 + b\tau x(m/2 - 1)} \quad . \tag{40}$$

Isocline curves for any assigned value of m pass through the singular point, $x_s = \beta$, $z_s = 0$, found earlier. All curves also pass through the origin, $x = 0$, $z = 0$, which is another singular point of more complicated nature than that found first. The slope of all isocline curves as they pass through the origin is the same, $dz/dx = a\tau$.

A set of isoclines is plotted in Fig. 8 upon axes of z and x. For this figure, positive numerical values of the parameters are assumed as $\beta = 1$, $b\tau = 1$, so that also $a\tau = 1$. According to the analysis based on variation of parameters, this value $a\tau = 1$ is just sufficient to lead to an oscillating solution about the singular point (x_s, z_s) with the amplitude of the oscillation remaining constant. Isoclines, calculated from Eq. (40), are plotted in Fig. 8 for several values of slope m, and line segments drawn through the isoclines have the corresponding slope.

Several solution curves are sketched in, always cutting the isoclines with the required slope. The predicted oscillation about the singular point is seen to occur. Since the solution curve is not a circle, but is distorted, the solution for x vs. t is not a simple sinusoid, although it is periodic. A solution curve coming from a large positive value of x bends so as to approach the z-axis closely, but never to cross it. This effect comes about from the very large positive damping occurring when x approaches zero from the positive side, as has been discussed previously. Because of this action, a limit cycle appears. For the assumed values of $b\tau$ and β,

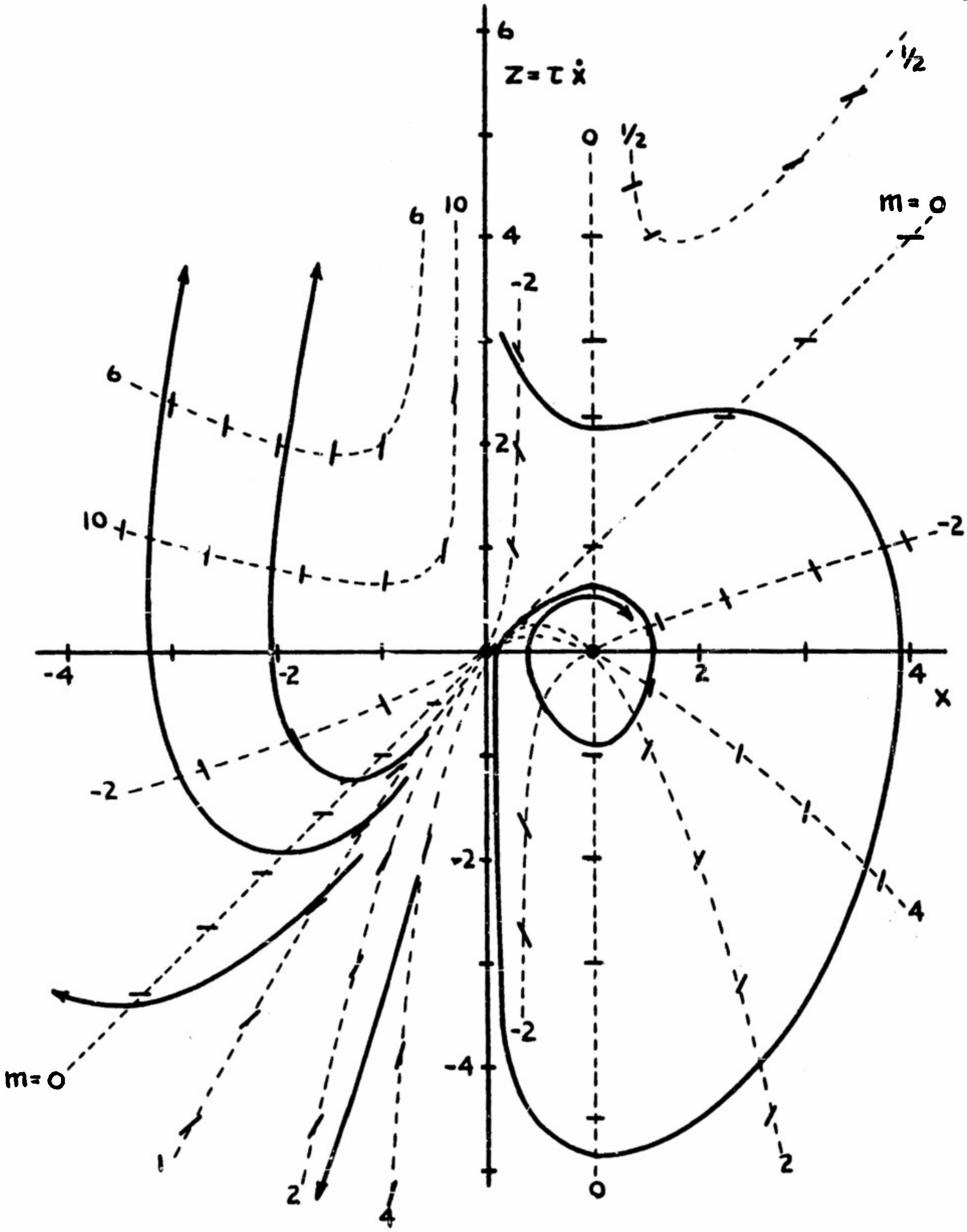


Fig. 8 Isocline construction for Eq. (9), $a\tau = 1$, $b\tau = 1$ $\beta = 1$.

a positive initial value of x leads to an oscillation about the singular point.

If x is initially negative, the solution curves sketched in Fig. 8 indicate that x always returns to zero with a value of z that goes to infinity. Presumably x will then pass through zero, with $\dot{x}$ infinite at that point, and become positive. As soon as x is positive, the solution curve spirals inward about the point $x_s = \beta$, $z_s = 0$, as before. It appears, therefore, that regardless of the initial algebraic sign of x, ultimately it becomes positive.

While the nature of the singularity at the origin cannot easily be studied so completely as was the singularity at $x_s = \beta$, $z_s = 0$, it is evident in this example that the origin is an unstable point. For positive values of x, the origin is similar to a saddle point; for negative values of x it is similar to an unstable nodal point.

III. 6. Approximate solution for negative x

Some of the qualitative aspects of the solution obtained for negative values of x can be found through the following approximate analysis. The differential equation is

$$(\tau^2/2)\,\ddot{x} - \tau\dot{x} + \dot{x}/bx + x = a/b \qquad (41)$$

which is merely Eq. (9) with the coefficients rearranged.

If $\tau = 0$, this equation reduces to Eq. (4), having the exact solution given in Eq. (7). This latter equation, rewritten here for convenience is

$$x_1 = \left[g + h\,\exp(-at)\right]^{-1} \qquad (42)$$

where $g \equiv b/a$ and $h \equiv (x_o^{-1} - g)$, with $x = x_o$ at $t = 0$.

If $\tau > 0$, but is small, an approximate solution for x can be found by an iteration process. The solution for x_1, given by Eq. (42) for $\tau = 0$, can be used with Eq. (41) for $\tau > 0$, to give a new solution for x as follows. From Eq. (41) the new solution is

$$x_2 = x_1 + \tau\dot{x}_1 - (\tau^2/2)\,\ddot{x}_1 \qquad (43)$$

where x_2 is an approximate solution, valid for τ small. In writing this equation, x_1 is put in place of the third and fifth terms of Eq. (41), to which it is equivalent.

A tentative conclusion from the isocline plot of Fig. 8 was that if x initially is negative, it always returns to zero, even if initially $\dot{x}$ is negative. In order for x to return to zero, $\dot{x}$ must become positive. If $\dot{x}$ is initially negative, there must be some instant at which it passes through zero as it changes sign. Thus, the time at which $\dot{x} = 0$ is of interest. This time can be determined approximately, making use of Eq. (43). The time derivative of this equation, with the assumption that τ is small enough that the last term can be neglected, is

$$\dot{x}_2 = \dot{x}_1 + \tau\ddot{x}_1. \qquad (44)$$

Derivatives of x_1, found from Eq. (42) and used here, give

$$\dot{x}_2 = \frac{ah\,\exp(-at)\,[g(1 - a\tau) + h\,\exp(-at)(1 + a\tau)]}{[g + h\,\exp(-at)]^3} \qquad (45)$$

If $\dot{x}_2 = 0$, the requirement is that, approximately,

$$\begin{aligned} \exp(-at) &= -(g/h)(1 - 2a\tau) \\ &= (1 - 2a\tau)/(1 - a/bx_0). \end{aligned} \qquad (46)$$

If $\tau = 0$, the value of t from this equation is the same as that found earlier as the time for which x, starting with a negative value, goes to minus infinity. If τ is made slightly greater than zero, the time for $\dot{x}_2 = 0$, given by Eq. (46), is also made larger.

The rate at which x changes, for τ just greater than zero also can be estimated. If $\tau = 0$ and if the magnitude of x is small enough, it is given approximately by

$$x_1 = x_0 \exp(at). \tag{47}$$

This equation results from Eq. (42) if $|x_0| \ll a/b$. This value of x_1 used in Eq. (43) leads to

$$(x_2 - x_1)/x_1 = a\tau(1 - a\tau/2). \tag{48}$$

This equation gives the difference between solution x_1, applying for $\tau = 0$, and solution x_2 applying for $\tau > 0$, but small. Since x_1 and x_2 must have the same algebraic sign, the left side of Eq. (48) is positive if $|x_2| > |x_1|$, and negative if $|x_2| < |x_1|$. It is evident from the right side of Eq. (48) that the fraction is positive, and thus $|x_2| > |x_1|$, for $0 < a\tau < 2$; otherwise $|x_2| < |x_1|$. These conclusions apply only so long as the magnitude of x remains small enough.

The result of this discussion can be summarized in Fig. 9, in which x is plotted against t, with parameters a and b held constant. The initial values for both $x = x_0$ and $\dot{x} = \dot{x}_0$ are negative, and both x_0 and $\dot{x}_0$ are assumed the same for all curves. The indication is that unless $\tau = 0$, the curves for x always return to zero, and that the point where $\dot{x} = 0$ occurs at later times as τ is increased. If $0 < a\tau < 2$, the magnitude $|x|$ initially increases faster than for $\tau = 0$. If $a\tau > 2$, the magnitude $|x|$ initially increases more slowly than for $\tau = 0$. The critical value, $a\tau = 2$, separating these kinds of solutions is only approximate because of the crude method by which it was obtained.

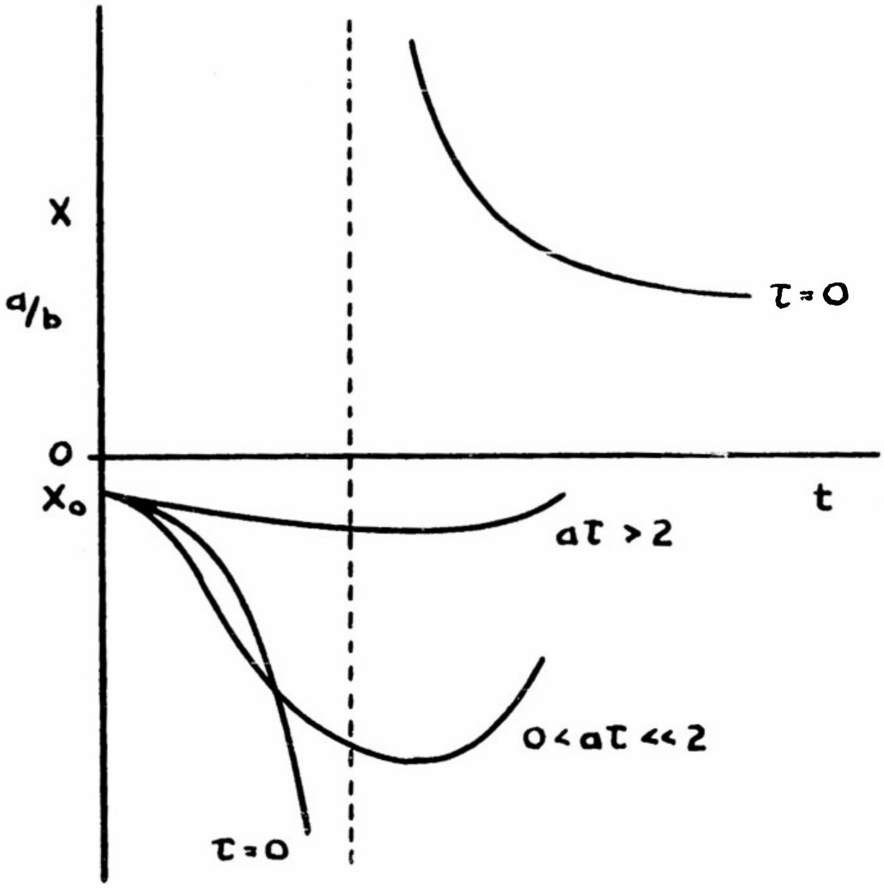


Fig. 9. Qualitative nature of solution for Eq. (9) with $X < 0$.

III. 7. Solution by analog computer

The differential equation, Eq. (9), was set up on a Reeves Analog Computer, using the system diagrammed in Fig. 10. Control of the parameters a, β ($= a/b$), and τ is possible, as shown. The additional inverting amplifier must be used when negative values of x are being studied, so as to give the correct algebraic sign to the quotient term. The computer gives accurate results so long as x and its derivatives are not too large, and so long as x is not too small. If x approaches zero, the division in the nonlinear term is inaccurate.

Some typical curves representing solutions for the equation are shown in the accompanying figures. In Figs. 11 and 12, a plot of x against t is shown. In both these figures, $\beta = 20$; in Fig. 11, τ is varied keeping a constant, while in Fig. 12, a is varied keeping τ constant. Initial conditions in both cases are $x_o = +\beta/4 = 5$, $\dot{x}_o = y_o = 0$. When the product $a\tau$ is unity, an oscillation about $x = \beta$ occurs, with its amplitude decaying slowly. The approximate analysis of Sec. III. 3. predicts a limit cycle for this value of $a\tau$, but with very small amplitude. A limit cycle of small amplitude may actually occur; it is difficult to decide from the computer solution. The period for the solution with $a\tau = 1$ is quite close to that predicted, $T = 2^{1/2}\pi\tau$.

If product $a\tau$ exceeds unity, the limit cycle with nonsinusoidal waveform becomes apparent. The waveform is similar to that predicted in Sec. III. 4. If product $a\tau$ is small, the final value of x is approached monotonically. Curves of $x = x_o \exp(at)$ and of x, as given by Eq. (7) for $\tau = 0$, are plotted in Fig. 11. These are limiting curves as τ varies between zero and infinity.

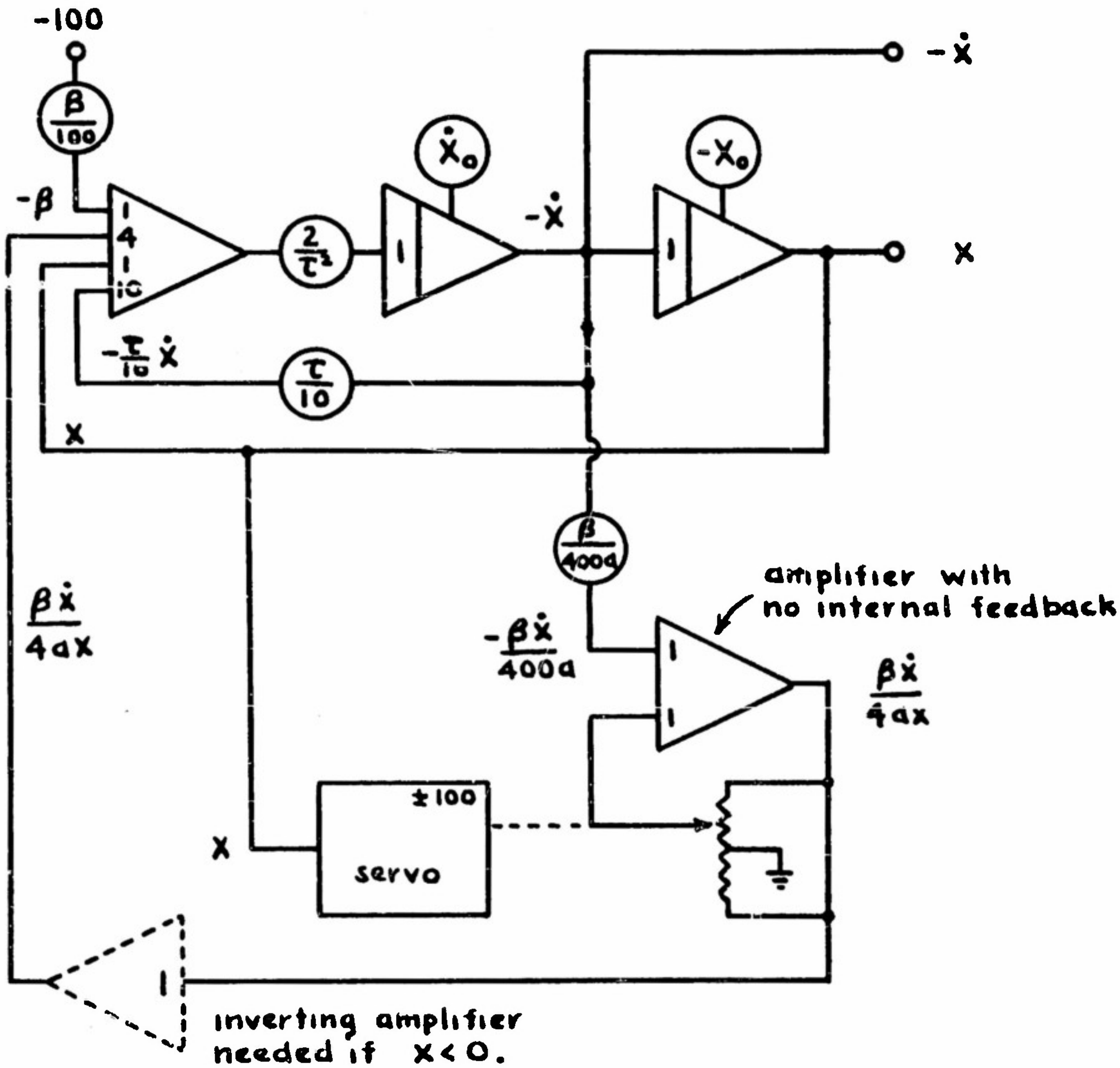


Fig. 10. Analog computer setup for solving Eq. (9).

Captions for Figs. 11-15

These figures are all solutions for Eq. (9) obtained with the analog computer set up of Fig. 10. The conditions for the figures are as follows.

11. $a = 1/2$, $\beta = 20$, τ as indicated, $x_o = 5$, $\dot{x}_o = 0$

12. $\tau = 2$, $\beta = 20$, a adjusted to give value for $a\tau$ as indicated, $x_o = 5$, $\dot{x}_o = 0$

13. $a = 1/2$, $\beta = 20$, τ adjusted to give value for $a\tau$ as indicated, x_o set to value indicated by circle, $\dot{x}_o = 0$

14. $a = 1/10$, $\beta = 10$, τ as indicated, $x_o = -5$, $\dot{x}_o = -0.75$

15. $a = 1/10$, $\beta = 10$, τ as indicated, initial conditions as indicated by circle

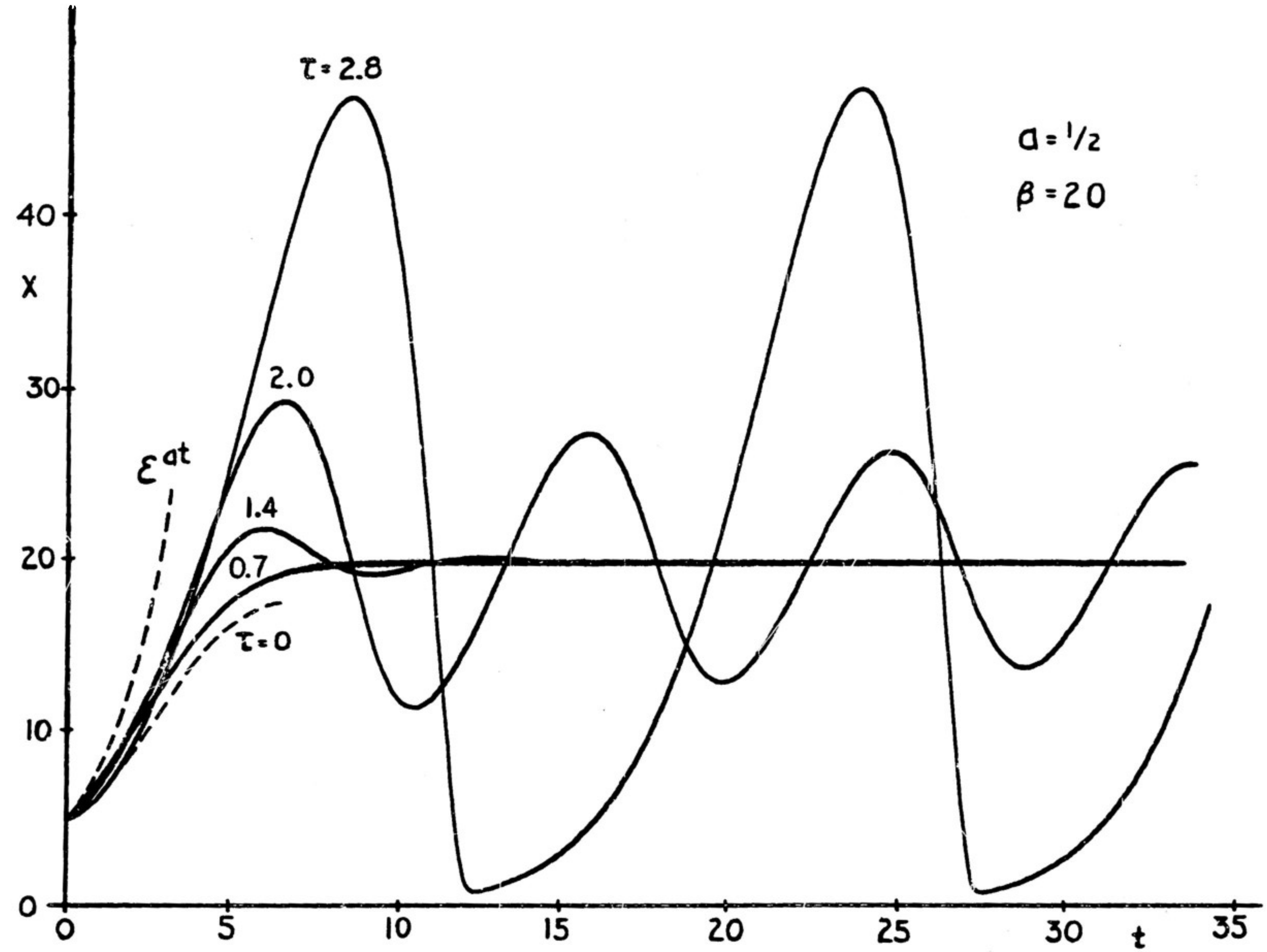


Fig. 11

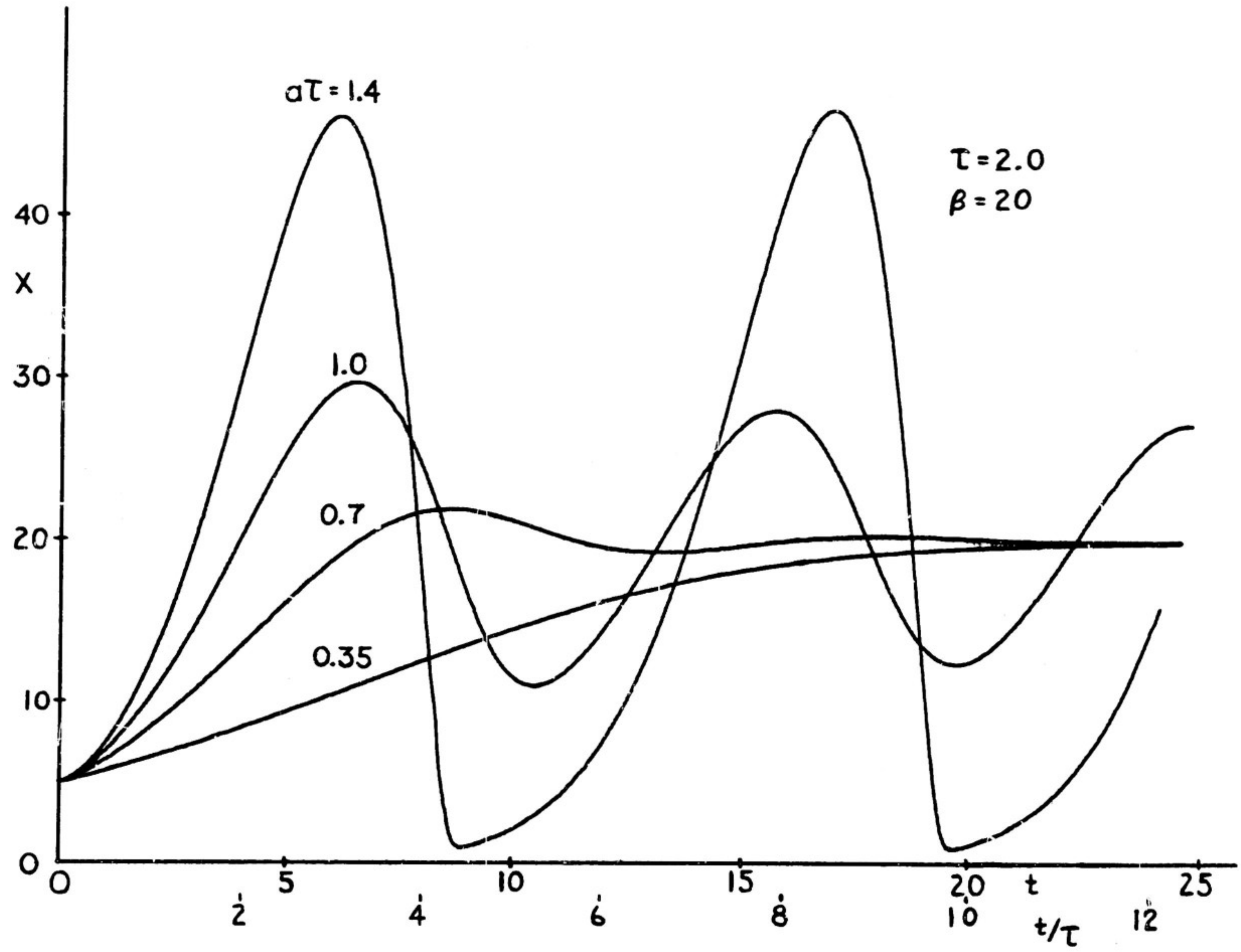
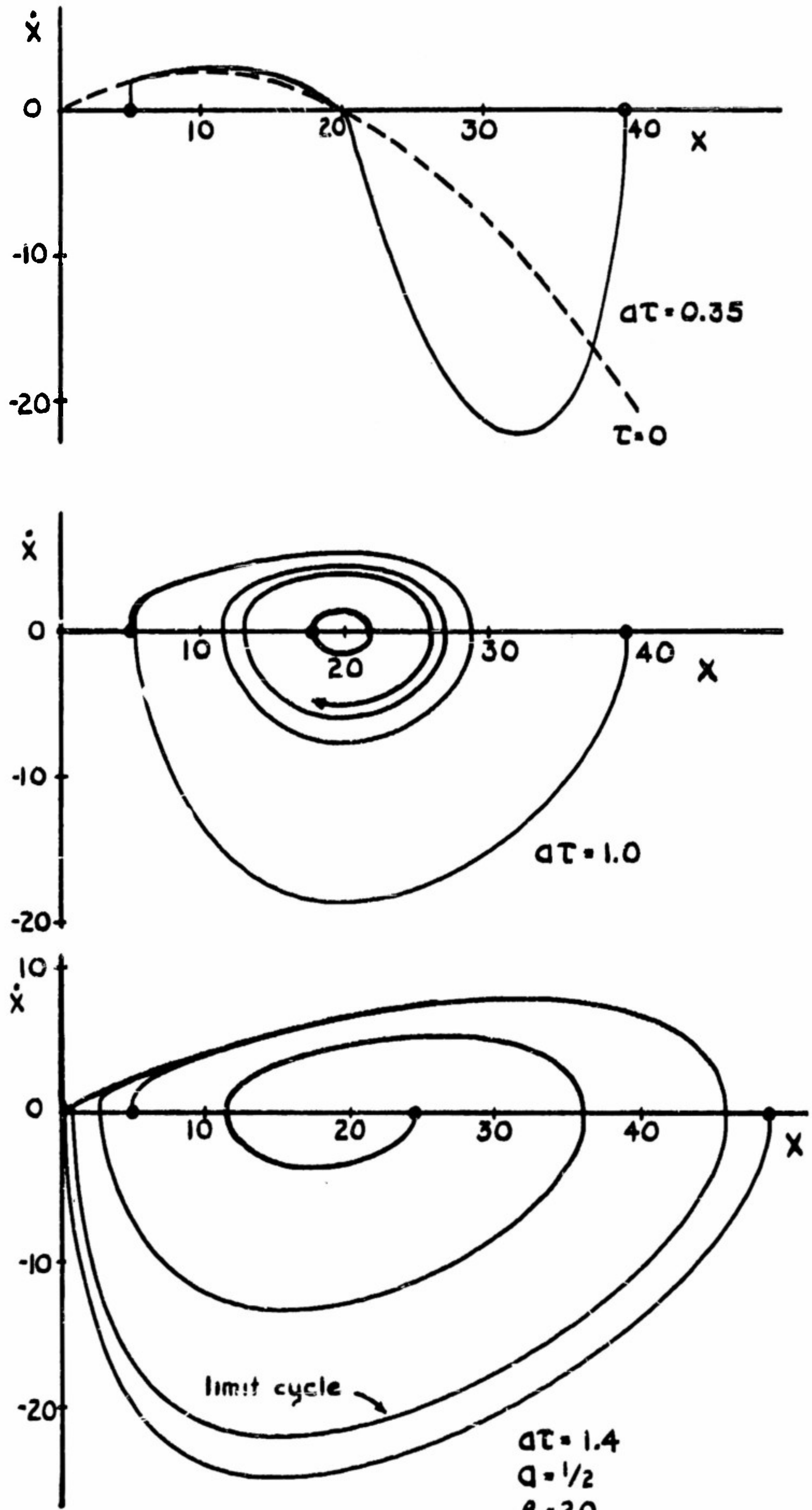


Fig. 12



ẋ
0
10
20
30
40
x
-10
-20
aτ = 0.35
τ = 0
ẋ
0
10
20
30
40
x
-10
-20
aτ = 1.0
10
ẋ
0
10
20
30
40
x
-10
-20
limit cycle
aτ = 1.4
a = 1/2
β = 20

Fig. 13

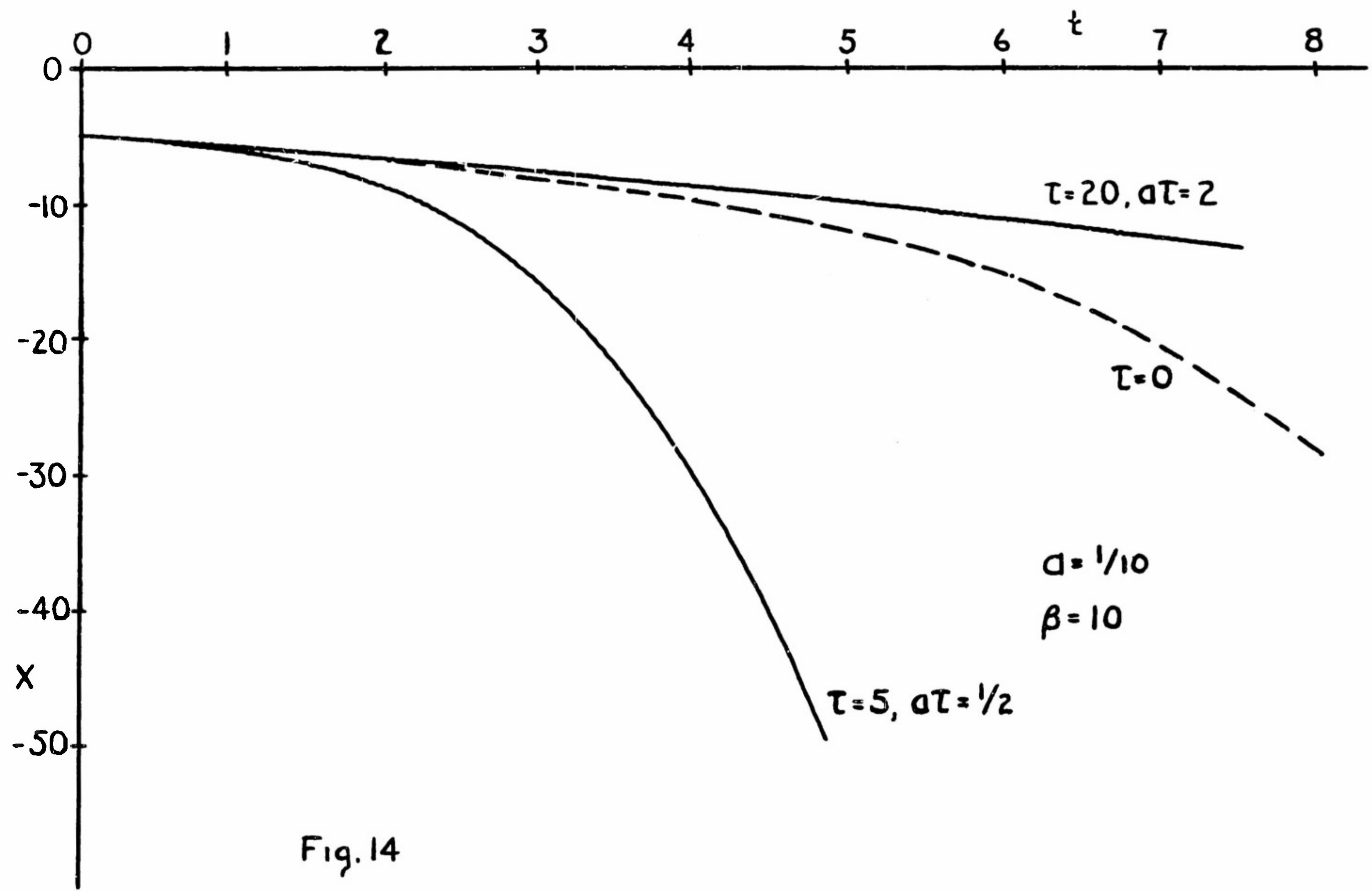
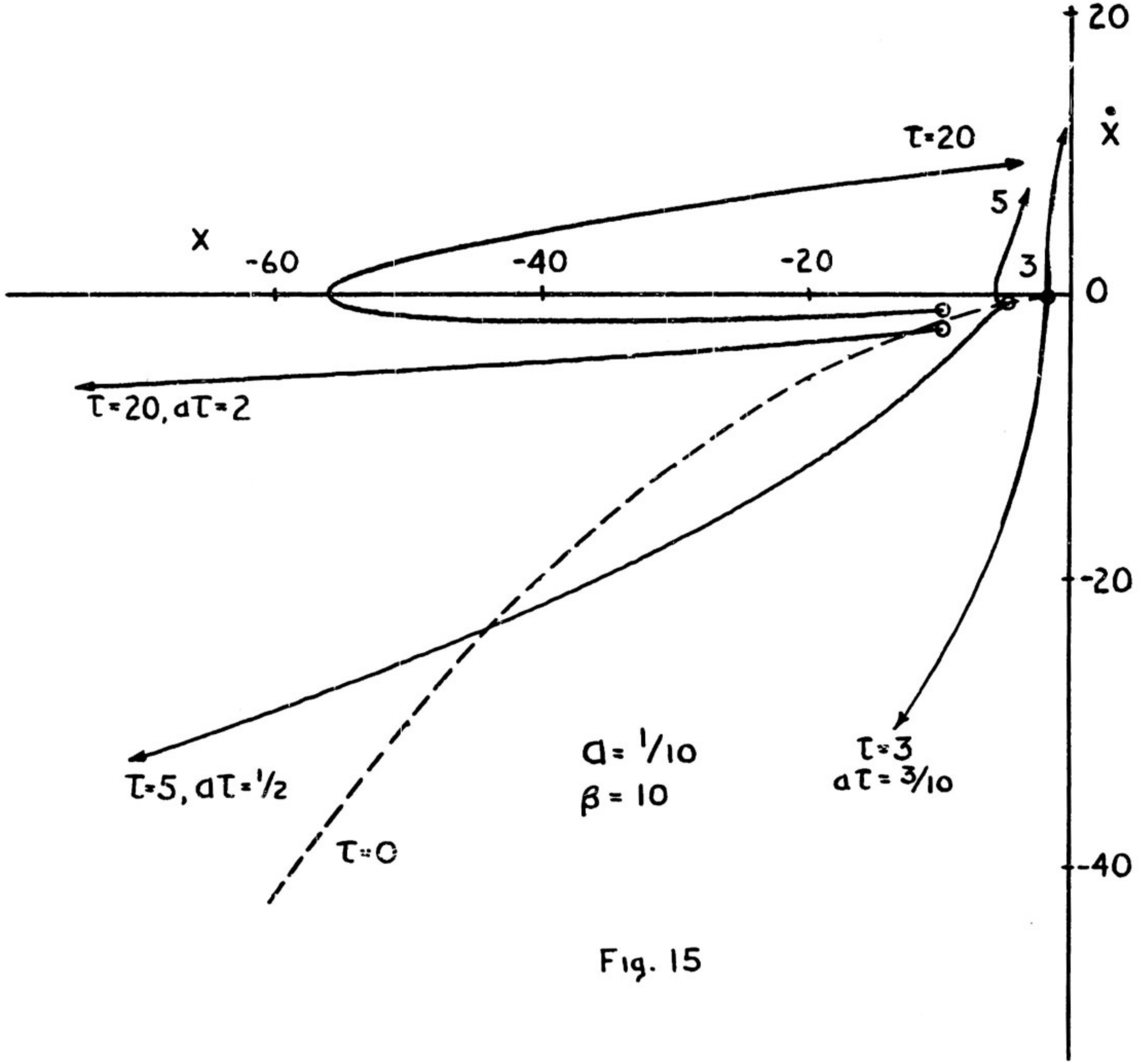


Fig. 14



20
$\dot{x}$
0
-20
-40
x
-60
-40
-20
3
5
τ=20
τ=20, aτ=2
τ=5, aτ=1/2
τ=0
τ=3
aτ=3/10
a= 1/10
β = 10

Fig. 15

In Fig. 13 are shown several phase-plane curves for y plotted against x. On each diagram are shown solution curves for several initial values of x_o, always with $y_o = 0$. The initial points are indicated with circles. The limit cycle for $a\tau = 1.4$ is evident, and a limit cycle of small amplitude may occur for $a\tau = 1$. The solution curve for $\tau = 0$, given by Eq. (4), is plotted for comparison on the diagram for $a\tau = 0.35$. Here, y jumps quickly from its initial zero value to a value near that which would have to exist with the case of $\tau = 0$.

The phase-plane curve obtained here for $a\tau = 1$ has a different vertical scale from that of Fig. 8, since the vertical coordinates are z and y, respectively. Otherwise the curves are very similar.

In Fig. 14, a plot of x against t is shown, with the initial value x_o being negative. Parameters were chosen as $a = 1/10$, $\beta = 10$, so that $b = 1/100$. The initial value of x is $x_o = -5$. A curve, calculated from Eq. (7) for $\tau = 0$ is shown for comparison. The initial slope of this curve, $y = -0.75$, was used as an initial condition in obtaining the other curves also. The curve for $\tau = 5$, $a\tau = 0.5$, is seen to depart from zero more rapidly than the curve for $\tau = 0$; the curve for $\tau = 20$, $a\tau = 2$, leaves zero more slowly. Phase-plane curves of y plotted against x are shown in Fig. 15, again for negative values of x. Once more, the curve for $\tau = 0$, calculated from Eq. (4), is shown for comparison. If τ is small, the solution curve either rises or falls abruptly, dependent upon whether the initial point is just above or below the curve for $\tau = 0$. If τ is larger, a slower change occurs, but the return to $x = 0$ is evident. The curves of Figs. 14 and 15 agree with the qualitative conclusions of Sec. III.6.

IV. Differential-Difference Equation

IV. 1. Comparison with differential equation

The original nonlinear differential-difference equation, Eq. (3), was replaced by a nonlinear differential equation, Eq. (9), for the analysis of Sec. III. The differential equation was obtained by replacing the difference term in the original equation by the first three terms of a Taylor's series, Eq. (8). The two equations should yield similar solutions so long as the Taylor's series approximates the difference term well. The approximation is good provided the time delay τ is small enough that the terms omitted from the series are small compared with those retained. In particular, it is necessary that $(\tau^3/6)\, d^3x/dt^3$ be much smaller than $(\tau^2/2)\, d^2x/dt^2$, and so on. This inequality holds if τ is sufficiently small. If τ is larger, it is unlikely that the approximation is valid. Solutions for the two equations then would be expected to differ by significant amounts.

An estimate of the magnitude of the terms can be found from the approximate solutions of Eqs. (36) and (37). If product $\alpha\tau$ is slightly larger than unity, approximately $x = \beta(1 + \sin \alpha t)$. Then $\ddot{x} = -\beta\alpha^2 \sin \alpha t$ and $\dddot{x} = -\beta\alpha^3 \cos \alpha t$ so that $(\tau^3/6)\dddot{x} = 2^{1/2}\beta/3$ and $(\tau^2/2)\,\ddot{x} = \beta$, where only the amplitudes appear in the last two relations. Thus, the first term omitted in the series has an amplitude about one half that of the last term retained, and the approximation is relatively poor. If product $\alpha\tau$ is larger, the solution changes more abruptly with time, and higher-order derivatives are larger. The approximation could be expected to be even less accurate.

Furthermore, the differential-difference equation is equivalent to a differential equation of infinite order, which may have an infinity of modes of oscillation. The approximate differential equation, of second order, can have only a single mode of oscillation. It is not difficult to see how the various modes for the difference equation can arise.

The analysis of the differential equation has predicted that if product $a\tau$ exceeds unity, a limit cycle appears, leading to an oscillation of period T as shown in Fig. 16. This solution can be assumed to apply to the differential-difference equation, also. According to this latter equation, the value of x at some time t depends, in part, upon its value at the earlier time $(t - \tau)$. If for example, the time $(t - \tau)$ is zero, time t at which x is determined, can be $t = \tau$, $(\tau + T)$, $(\tau + 2T)$, $\cdot \cdot \cdot \cdot (\tau + nT)$. For any of these times, the same result must be obtained for x, because of the periodicity of the solution. The same steady-state solution results if the original delay time is increased by any integral multiple of the period. The solution of Fig. 16, then, can be obtained for fixed values of parameters a and b, and an infinity of values for τ. The period and waveform remained unchanged.

If all three parameters, a, b, and τ, are fixed, a variety of steady-state solutions also may occur. An example is shown in Fig. 17, where the waveforms are merely sketched roughly. If the product $a\tau$ is large enough, a violent oscillation of long period, $T_1 = \tau/k$, may occur, where k is a constant. If $a\tau$ is just larger than unity, the period of the oscillation has been shown to be $T = 2^{1/2}\pi\tau$, so that $k = (2^{1/2}\pi)^{-1} = (4.44)^{-1}$. For larger values of $a\tau$, k will be smaller, but the effective time delay remains in the order of 1/4

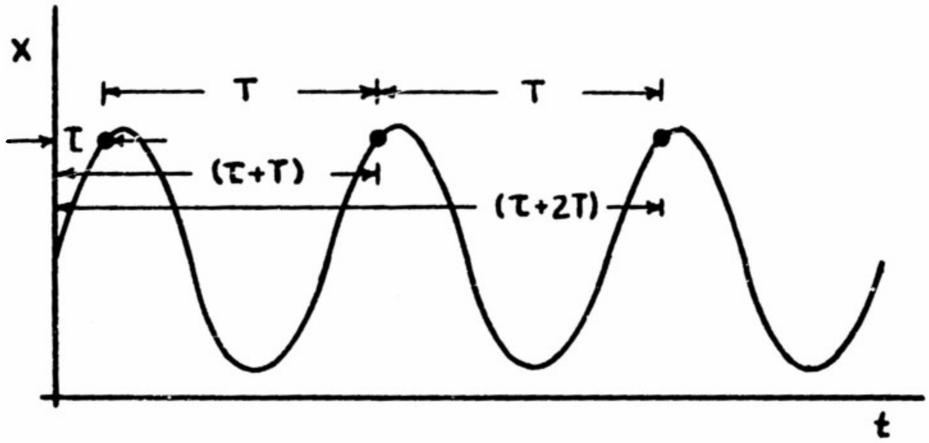


Fig. 16. Periodic solution obtained with different delay times.

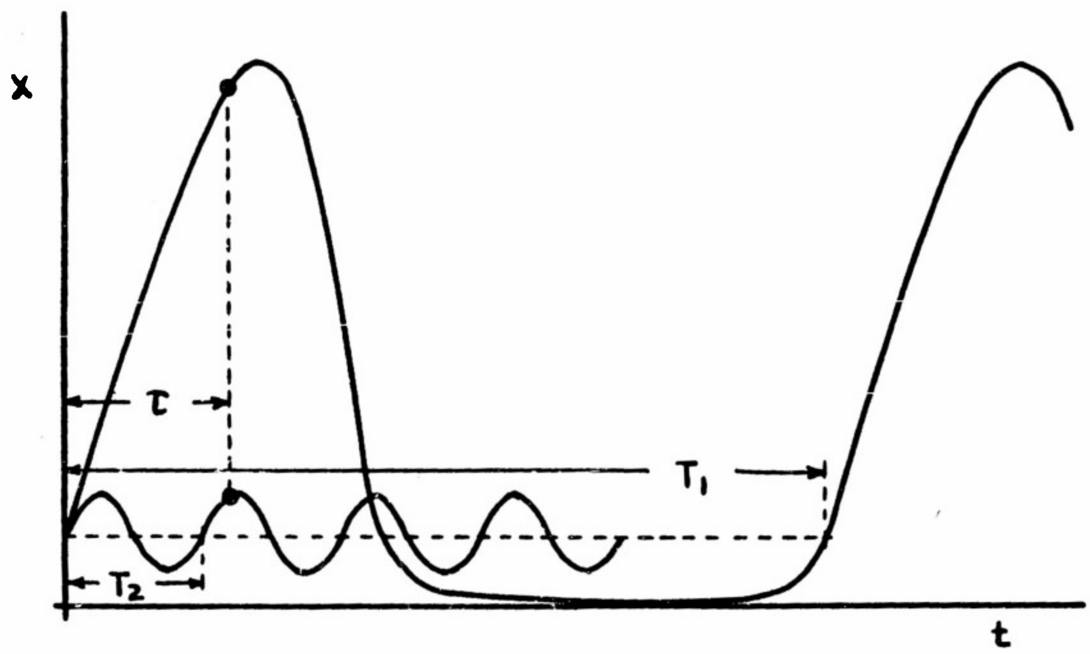


Fig. 17. Two different periodic solutions obtained with the same parameters.

of a period. For an initial value of $a\tau$ sufficiently large, another steady-state oscillation may occur, with period $T_2 = \tau/(1 + k)$. The effective time delay is then in the order of 5/4 of a period, and the period of the second oscillation is about 1/5 that of the first. The waveform of the second oscillation is determined by the product $a\tau'$, where the value of τ' is the least delay that could give the observed oscillation, or about 1/5 the actual value of τ. Thus, the waveform of the oscillation of shorter period depends upon the quantity $a\tau/5$, approximately, and if there is to be a steady-state oscillation this quantity must be large enough to lead to a limit cycle, that is, somewhat larger than unity. The oscillation of longer period will have its waveform dependent upon $a\tau$, which will be greater than five, and the waveform will involve large peaks and violent changes.

If product $a\tau$ were sufficiently large, a whole sequence of steady-state oscillations of this sort might be obtained, each having a different period and a different waveform. Suitable and rather special initial conditions would be required to start such oscillations, however. It is likely that only the oscillation of longest period will actually occur in any physical system to which the equation applies.

IV. 2. Solution by analog computer

The differential-difference equation is difficult to set up on a simple analog computer because of the necessity for introducing the time delay τ. This delay time would have to be reasonably long in comparison with the period of oscillation of the highest frequency that can be handled by the computer. It is not easy to set up a system with the available components so as to obtain the necessary delay.

In order to avoid the need of introducing a time delay in the computer itself, a step-by-step computation was used. The computer was used to solve the equation

$$dx(t)/dt = \left[a - b\, x(t_o + \Delta t/2 - \tau)\right] x(t) \qquad (49)$$

in place of Eq. (3). In Eq. (49), Δt is the time interval during which computation occurs and t_o is the time at the beginning of the interval. In the bracket, a constant value for x is used, evaluated at the delay time τ earlier than the average of the time during the interval. If the time interval is chosen small enough, this procedure should lead to accurate results. It suffers, of course, from the usual errors of step-by-step computations.

The computer was set up as shown in Fig. 18. The procedure is to put into the multiplying circuit of the computer the value of $x(t_o + \Delta t/2 - \tau)$. The value of $x(t_o)$ is put into the integrator as an initial condition. The computer is allowed to run for the time interval Δt, after which the value of $x(t_o + \Delta t)$ is read. This new value of x is then used for new initial conditions with the next time interval, and the procedure is repeated. In this way successive points for a curve of x versus t can be obtained.

A family of curves found in this way is shown in Fig. 19, where x is always positive. Numerical values for the parameters were $\tau = 18$, $\beta = 20$, and $\Delta t = 4$. Parameter a was adjusted to give product $a\tau$ the indicated value. The initial value of x at $t = 0$ was taken as $x_o = 2$. It was assumed further than x had this same value during the four intervals Δt preceding $t = 0$. This assumption gives x the constant value, $x = 2$, during the time $-16 \leq t \leq 0$, after which x begins to change. With a value of x as small as this,

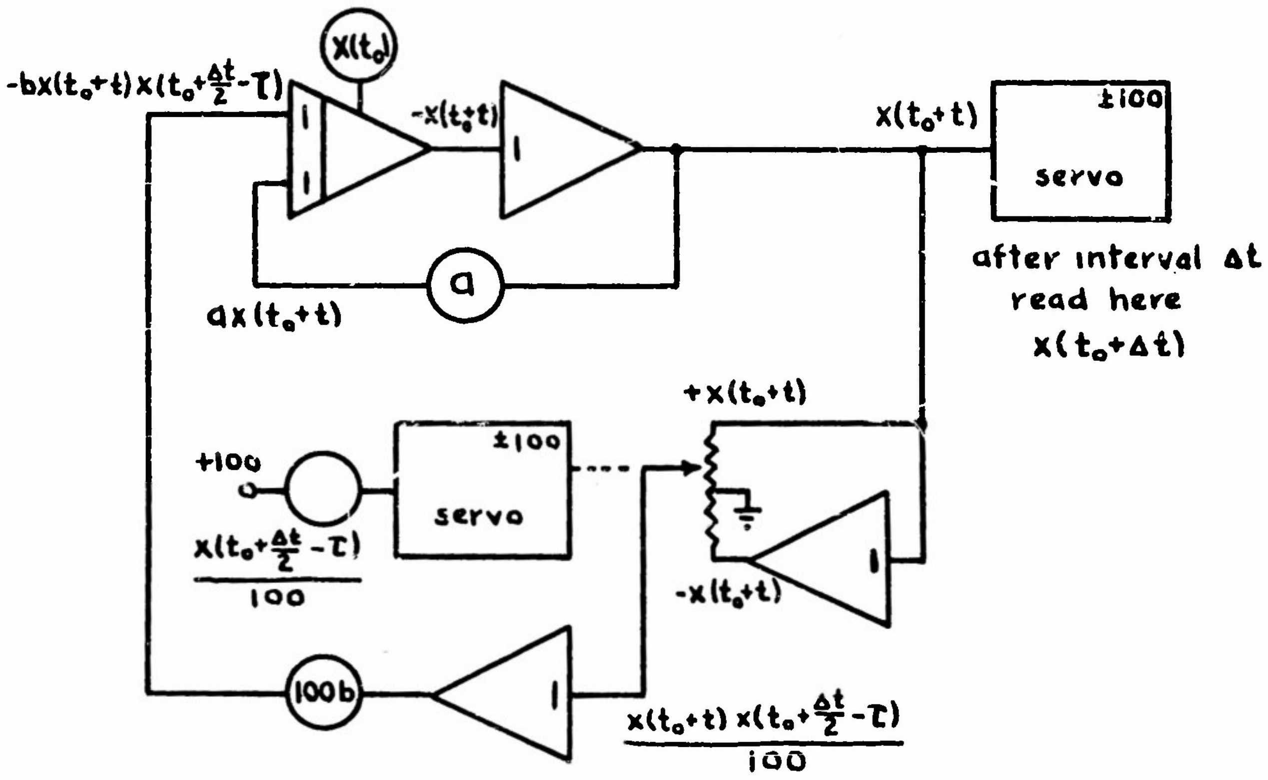


Fig. 18. Analog computer setup for solving Eq. (49)

Captions for Figs. 19-20

These figures are solutions for Eq. (3) obtained with the analog computer set up of Fig. 18. The conditions for the figures are as follows.

19. $\tau = 18$, $\beta = 20$, $\Delta t = 4$, a adjusted to give value for $a\tau$ as indicated, $x = 2$ for $-16 \leq t \leq 0$

20. $\tau = 18$, $\beta = 20$, $\Delta t = 4$, a adjusted to give value for $a\tau$ as indicated, $x = -2$ for $-16 \leq t \leq 0$

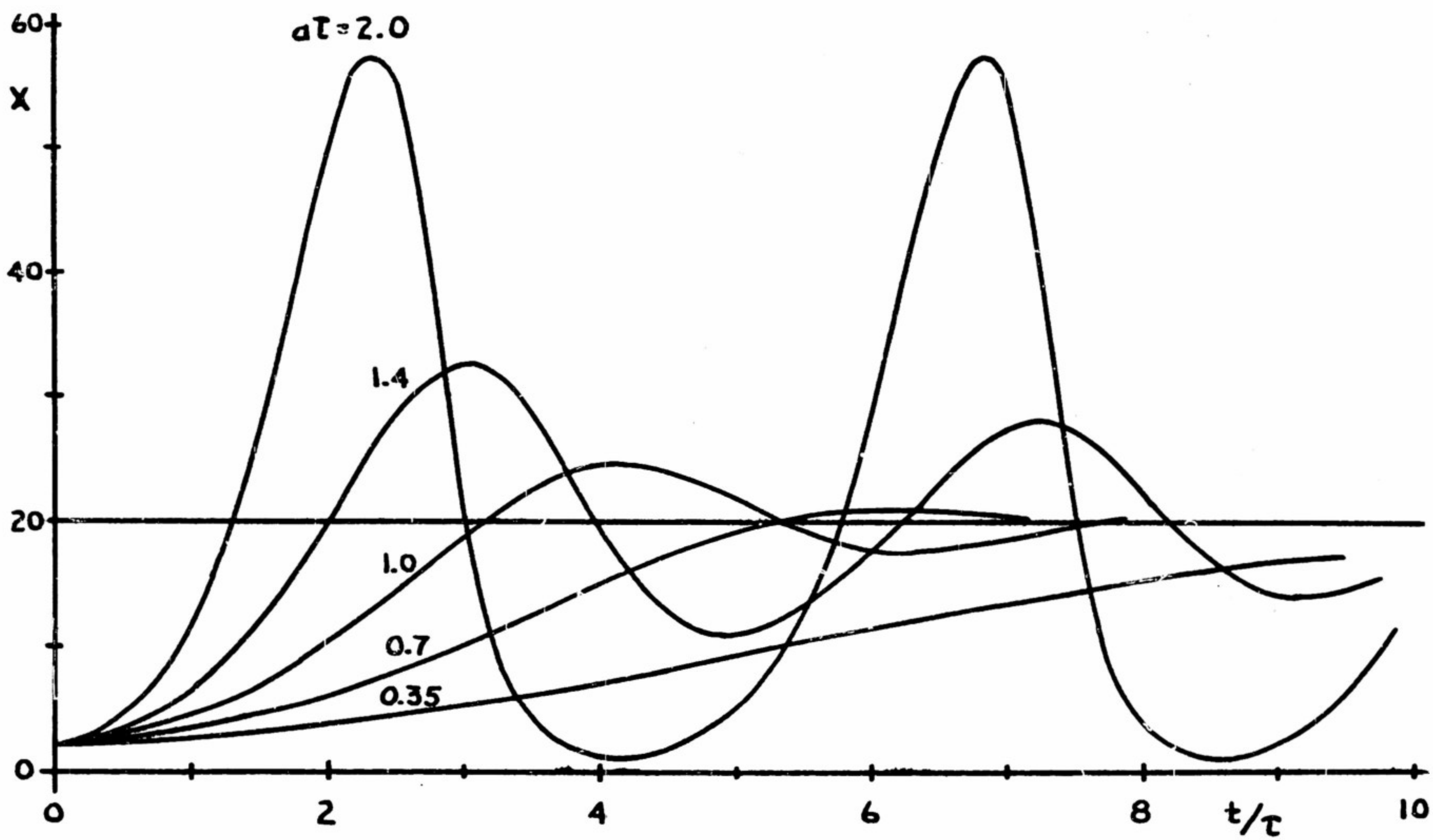


Fig. 19.

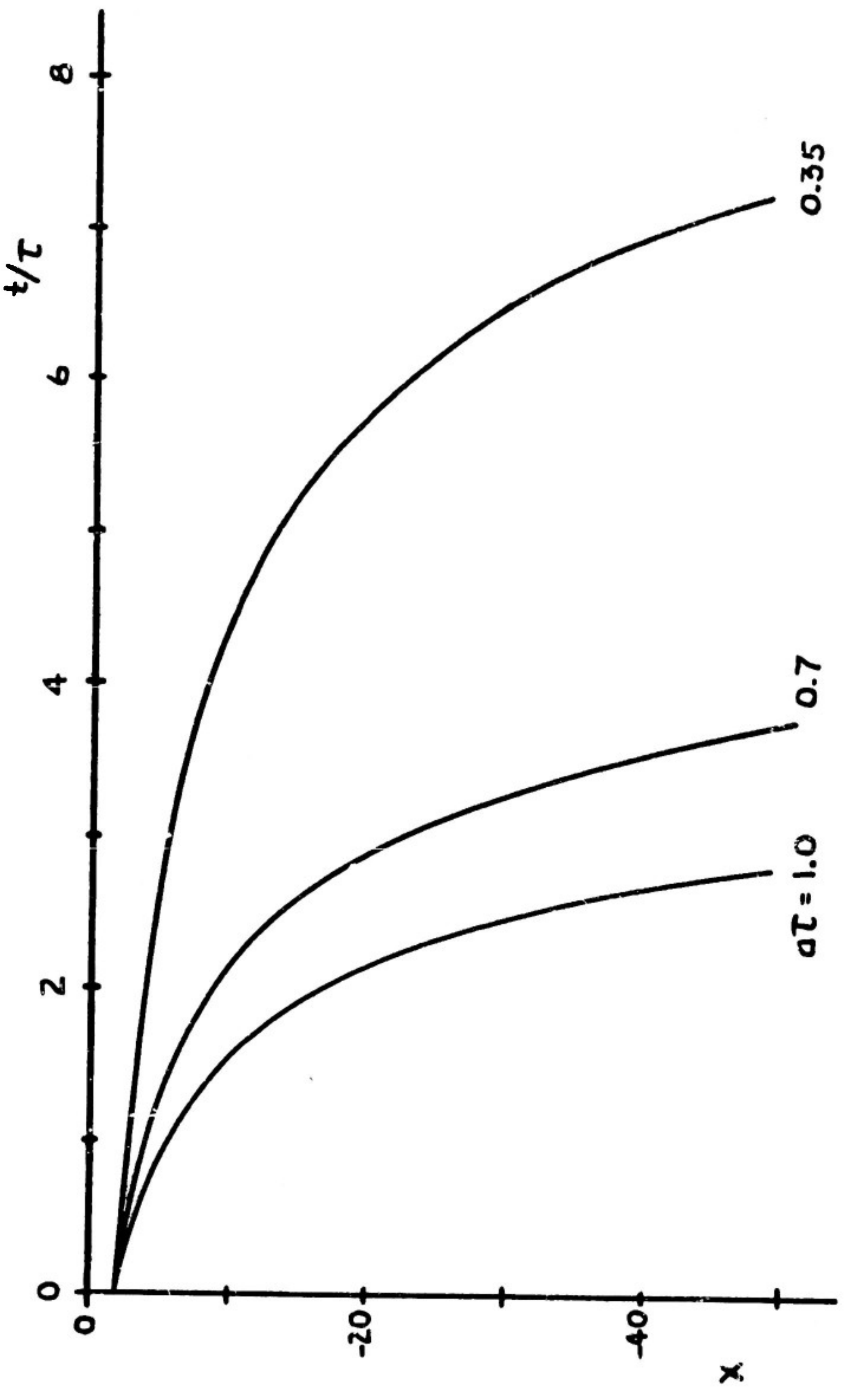


Fig. 20.

in comparison with $\beta = 20$, the exact value used in starting the computation does not influence the resulting solution curve very much.

The curves of Fig. 19 for the differential-difference equation are similar qualitatively to those of Fig. 12 for the differential equation. The period of the oscillation is similar for a given value of product $a\tau$. However, the minimum value of $a\tau$ leading to a limit cycle for the differential equation, $a\tau \doteq 1$, is less than that, $a\tau \doteq 1.6$, needed for the differential-difference equation. A difference of this sort is not unexpected, however, because of the approximation that was used in getting the differential equation. In the appendix is given a discussion which might indicate the limit cycle would first appear near $a\tau = \pi/2$.

Another family of curves, with x negative, is shown in Fig. 20. The same numerical values of the parameters were used in obtaining Fig. 20 as in Fig. 19, except for the initial value of x. For Fig. 20 it was assumed that $x = -2$ for the time $-16 \leq t \leq 0$. The curves of Fig. 20 show the expected shape, with x becoming an increasingly large negative quantity as t increases. The rate of increase depends upon the value of parameter a. The monotonic increase of the curves of Fig. 20 is again different from the solutions for the differential equation, shown in Fig. 14, where x always returns to zero. Again, this results because of a basic difference in the two equations.

V. Conclusion

The differential-difference equation has been approached in a variety of ways, and a number of approximate solutions for it have been obtained. Most complete information comes from the differential equation that is approximately equivalent. Information about the solution is summarized in Tables 1 and 2.

Table 1

<u>Differential-Difference Equation</u>

1. Form: $$dx(t)/dt = \left[a - b\, x(t - \tau)\right] x(t) \tag{3}$$

   a, b, τ are positive real constants

2. Equilibrium conditions:

   $x(t) = x(t - \tau) = 0$, unstable

   $x(t) = x(t - \tau) = a/b$, stable with, perhaps, a limit cycle

3. Algebraic sign of solution: solution cannot change sign

4. Exact solution for $\tau = 0$:

   $$x = \left[b/a + (x_o^{-1} - b/a) \exp(-at)\right]^{-1} \tag{7}$$

5. Approximate solution for $x > 0$: qualitatively similar to solutions for the differential equation, Eq. (9). See Fig. 19.

6. Probable value of $a\tau$ for oscillation, $x > 0$: $a\tau > 1/2$, this value is estimated

7. Probable value of $a\tau$ for a limit cycle, $x > 0$: $a\tau > \pi/2$

8. Solution for $x < 0$: solution always goes to $x = -\infty$. See Fig. 20.

Table 2

Differential Equation

1. Form: $\ddot{x} - \alpha^2\tau\dot{x} + \alpha^2 \dot{x}/bx + \alpha^2 x = \alpha^2\beta$ (9)

$\alpha^2 = 2/\tau^2$, $\beta = a/b$, where a, b, τ are positive real constants

2. Equilibrium conditions:

$x = 0$, $\dot{x} = 0$, unstable

$x = a/b$, $\dot{x} = 0$, stable with, perhaps, a limit cycle

3. Algebraic sign of solution: Starting with either sign, solution always becomes positive.

4. Exact solution for $\tau = 0$:

$$x = \left[b/a + (x_o^{-1} - b/a)\exp(-at)\right]^{-1} \qquad (7)$$

5. Approximate solution for $x > 0$:

For $4/5 < a\tau < 5/4$:

$$x = \beta + A(t)\sin(2^{1/2}t/\tau + \theta_o) \qquad (27)$$

$$A(t) = \left[q/p + (A_o^{-2} - q/p)\exp(-2pt)\right]^{-1/2} \qquad (25)$$

$$p = (a\tau^2)^{-1}(a\tau - 1), \qquad q = b^2/4a^3\tau^2$$

at $t = 0$, $A = A_o$, $\theta = \theta_o$.

In steady state, $1 < a\tau < 5/4$:

$$x = \beta + A\sin\alpha t - \alpha A^2/6b\beta^2 \sin 2\alpha t + \cdot\cdot\cdot\cdot \qquad (37)$$

See Fig. 12

6. Probable value of $a\tau$ for oscillation, $x > 0$:

$$a\tau > (2^{1/2} - 1) = 0.414$$

7. Probable value of $a\tau$ for a limit cycle, $x > 0$:

$$a\tau > 1$$

8. Solution for $x < 0$: solution always ultimately returns to $x = 0$ and becomes positive, with $\dot{x} = \infty$ at $x = 0$. Nature of solution depends upon parameters of equation and initial conditions. See Fig. 14.

Appendix A. Method of Equivalent Linearization

There is yet another method that is sometimes useful in finding an approximate solution for an equation that cannot be solved exactly. This is the method of equivalent linearization, sometimes referred to under the names of Kryloff and Bogoliuboff.[15,16] It can be applied to the equation under discussion here, which is

$$dx(t)/dt = \left[a - b\,x(t-\tau)\right] x(t). \tag{3}$$

Under some conditions this equation has been shown to have steady-state solutions, oscillating about a mean value.

The procedure is to assume a solution of the necessary form

$$x(t) = P + Q \sin \omega t \tag{A.1}$$

where P, Q, and ω are all constants to be determined. The assumed solution is then substituted into Eq. (3), giving

$$\begin{aligned} Q\omega \cos \omega t = {} & (aP - bP^2 - \tfrac{1}{2} bQ^2 \cos \omega\tau) \\ & + (aQ - bPQ - bPQ \cos \omega\tau) \sin \omega t \\ & + (bPQ \sin \omega\tau) \cos \omega t \\ & + \tfrac{1}{2} bQ^2 (\cos \omega\tau \cos 2\omega t + \sin \omega\tau \sin 2\omega t). \end{aligned}$$

It is argued that this equation must be valid for those components of zero frequency and the fundamental frequency, and terms of frequency higher than the fundamental are ignored. The following three relations are thus obtained.

$$\left.\begin{aligned} &\underline{\text{constant}}: && aP - bP^2 - \tfrac{1}{2} bQ^2 \cos \omega\tau = 0 \\ &\underline{\sin \omega t}: && aQ - bPQ - bPQ \cos \omega\tau = 0 \\ &\underline{\cos \omega t}: && Q\omega - bPQ \sin \omega\tau = 0 \end{aligned}\right\} . \tag{A.2}$$

These equations are not correct if $\tau = 0$, since then $\omega = 0$, and the term in $\cos 2\omega t$ becomes a constant and must be considered.

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15. N. Minorsky, ref. 7, Ch. XII

16. F. E. Bothwell, Econometrica, 20, 269, (1952)

The relations can be rearranged to give

$$\left.\begin{aligned} \sin \omega\tau &= 2a\omega/(a^2 + \omega^2) \\ \cos \omega\tau &= (a^2 - \omega^2)/(a^2 + \omega^2) \end{aligned}\right\} \tag{A.3}$$

$$P = (a^2 + \omega^2)/2ab \tag{A.4}$$

$$Q = 2^{1/2} P. \tag{A.5}$$

These equations must be satisfied simultaneously, and can be solved for the necessary values of ω, P, and Q.

It is evident from the transcendental form of Eq. (A.3) that an infinity of values of ω, the angular frequency of oscillation, are allowed. This is the result expected, since the original equation is equivalent to a differential equation of infinite order. However, the relation between P and Q, Eq. (A.5), cannot correspond to a correct solution for Eq. (3). It requires that Q exceed P, so that instantaneous values of x always change sign twice during a cycle of the oscillation, something which cannot occur. Furthermore, Eq. (A.4) indicates that the amplitude of oscillation is larger for a higher-frequency mode of oscillation, which seems to be unreasonable.

If product $a\tau = \pi/2$, Eq. (A.3) predicts that $\omega_1\tau = \pi/2 = 1.57$, and Eq. (A.4) predicts that $P_1 = a/b$. This result agrees fairly well with the discussion of Sec. III. 3 where for $a\tau$ near unity, it was found that $\omega\tau = 2^{1/2} = 1.41$ and the mean value of the solution is $\beta = a/b$. A second value of $\omega\tau$, allowed by Eq. (A.3) with $a\tau = \pi/2$, is about $\omega_2\tau = 9.1$ or approximately, $\omega_2 = 5.8\ \omega_1$. This result is also about what is expected, with the ratio of the first two frequencies of oscillation being in the order of five to one. The value of P for this second frequency, from Eq. (A.4), is predicted as $P_2 = 17\ a/b$, and this seems far too large. Furthermore, numerical

values found from Eqs. (A.3-5) do not appear reasonable for values of product $a\tau$ differing from $a\tau = \pi/2$.

The conclusion must be drawn that this method of solution yields results of some utility only for a narrow range of values of product $a\tau$ near $a\tau = \pi/2$. Most likely this is about the value of product $a\tau$ which first gives a limit cycle in the solution of the differential-difference equation. In general, however, the method gives results that seem quite unreasonable. Presumably this occurs because solutions for Eq. (3) are sufficiently non-sinusoidal that the assumed form of Eq. (A.1) is far from correct.

Appendix B. Approximate Solution for $a\tau \gg 1$

An exact solution for Eq. (3) with the time delay equal to zero was found in Sec. II. Solutions with values of product $a\tau$ in the order of unity were found in Sec. IV. It is of interest to explore the nature of the solution if $a\tau$ becomes very large.

It was shown in Secs. III and IV that if $a\tau$ is sufficiently large, the solution is periodic with the period T. The oscillations are nearly sinusoidal for $a\tau$ just large enough to give the periodic solution, and become violent relaxation oscillations for $a\tau$ very large. Always, however, the mean value, $\bar{x}$, of the solution is the same, $\bar{x} = a/b = \beta$. This relation can be shown as follows.[+]

Equation (3) can be written as

$$\frac{dx(t)/dt}{x(t)} = a - b\,x(t - \tau). \tag{B.1}$$

If the solution is periodic with period T, both sides of this equation can be averaged over a period to give the relation

$$\frac{1}{T}\int_0^T \frac{dx(t)}{x(t)} = \frac{1}{T}\int_0^T \left[a - b\,x(t - \tau)\right] dt. \tag{B.2}$$

Because of the periodicity, the integral on the left side of the equation is zero. Also, the mean value of the oscillating solution is

$$\bar{x} = (1/T)\int_0^T x(t)\,dt = (1/T)\int_0^T x(t - \tau)\,dt. \tag{B.3}$$

Thus, Eq. (B.2) becomes

$$0 = a - b\,\bar{x}$$

[+] This fact was observed by L. Onsager (ref. 3, p. 237). The derivation given here was suggested by P. M. Schultheiss.

and

$$\bar{x} = a/b = \beta. \tag{B.4}$$

This equation is valid for any periodic solution.

As product $a\tau$ is made large, a plot of the solution for Eq. (3) approximates the shape sketched in Fig. B.1, where the time origin is chosen at the point at which x crosses the value $x = \beta$ as it increases in magnitude. The rise from $x = \beta$ to the maximum value, x_{max}, takes place almost linearly. The drop from x_{max} to a very small value, nearly $x = 0$, takes place very abruptly. The solution stays at a small value most of the remainder of the period, and then begins to rise again rather suddenly. The shape of Fig. B.1 is considerably simplified, of course; the actual solution curve does not change value discontinuously.

At the time $t = 0$ in Fig. B.1, $x = \beta$, and the value of x at the earlier time, $(t - \tau) = (-\tau)$, is essentially zero. Thus, at $t = 0$, the slope of the solution curve is almost $dx(0)/dt = a\beta$, from Eq. (3). It is assumed that this slope is maintained over the inverval $0 \leq t \leq \tau$. At the end of the interval, $t = \tau$, and from Eq. (3), the slope must vanish, $dx(\tau)/dt = 0$, since $x(t - \tau) = \beta$. Thus, at $t = \tau$, x has its maximum value, $x_{max} = k\beta$, this relation defining the constant k. The combination of the known slope for the solution curve, maintained over the known time interval, allows the value of k to be determined as

$$dx/dt = a\beta = (k - 1)\beta/\tau$$

or

$$k = (1 + a\tau). \tag{B.5}$$

Thus, the maximum value of x is

$$x_{max} = (1 + a\tau)\beta. \tag{B.6}$$

Since it was shown in Eq. (B.4) that the mean value of the periodic solution is constant, the positive and negative areas between the solution curve and $x = \bar{x} = \beta$ must be the same in magnitude. Thus, the relation can be written

$$\tau(k - 1)\beta/2 = (T - \tau)\beta$$

or the period is

$$T = (k + 1)\tau/2 = (1 + a\tau/2)\tau. \tag{B.7}$$

Equations (B.6) and (B.7) can be used for sketching the approximate shape of the solution curve, provided product $a\tau$ is known and is sufficiently large. Probably the maximum value of x, given by Eq. (B.6), is reasonably accurate for $a\tau \geq 2$. The period, given by Eq. (B.7), probably is not very accurate unless $a\tau \geq 10$, however. The poor accuracy in this latter equation results from the assumption of discontinuous jumps in the solution, and these approximate the actual solution well only if $a\tau$ is quite large.

An estimate of the ratios (x_{max}/β) and (T/τ) for periodic solutions of Eq. (3) is shown in Fig. B.2. These estimates are based on the solutions found in Secs. III and IV for $a\tau$ near two, and upon the equations just developed for $a\tau$ very large. Because of the wide range in values of x which occurs when $a\tau$ is large, it has not been possible to check these estimates quantitatively upon the analog computer.

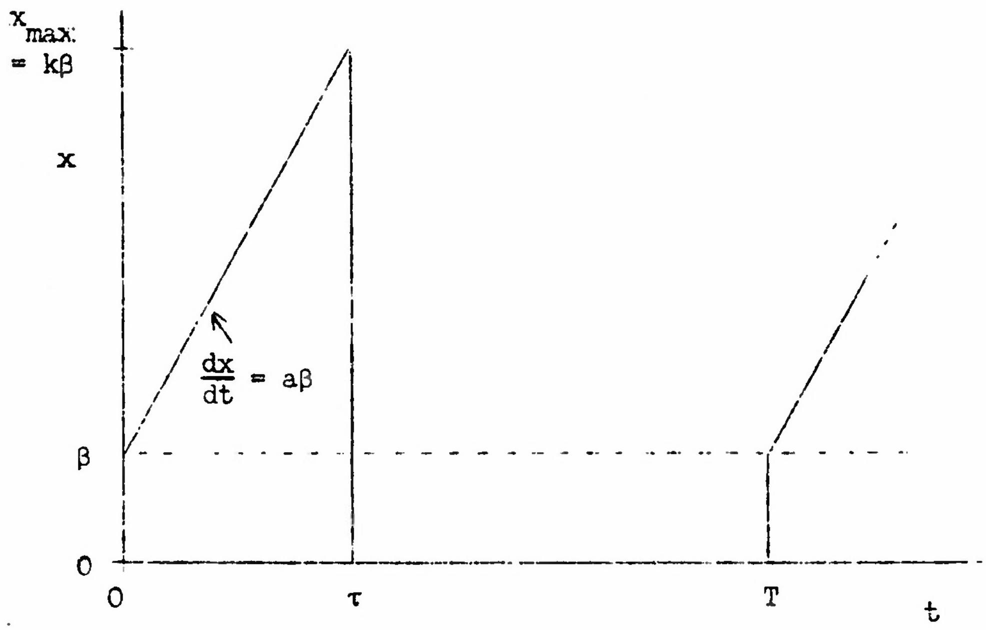

Fig. B.1 Approximate solution for Eq. (3) with $a\tau$ very large.

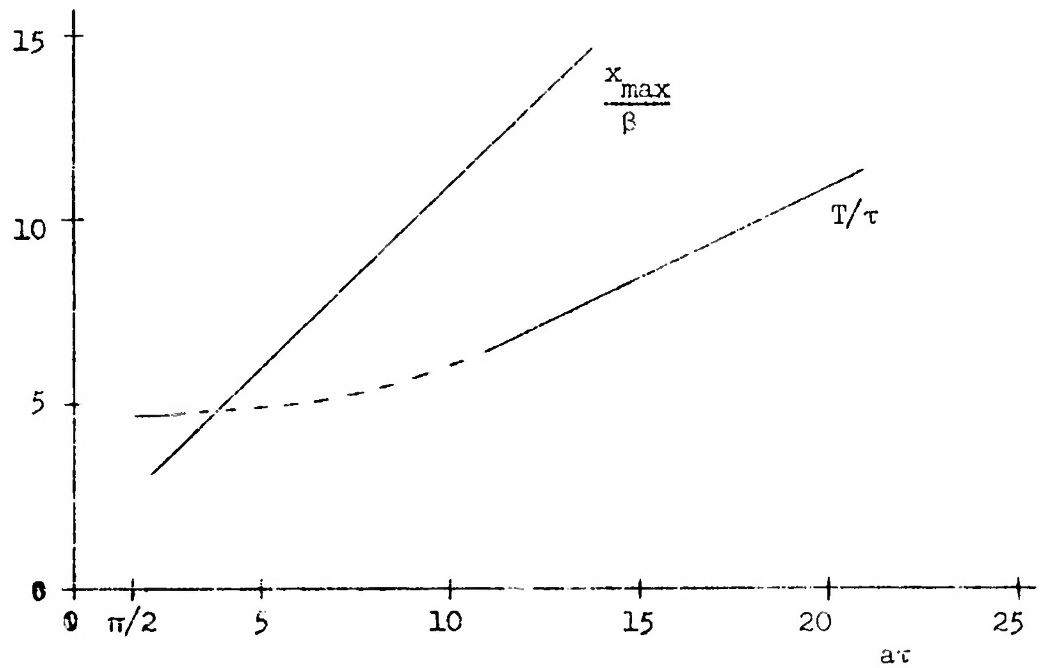

Fig. B.2 Estimated period and maximum value of solution for Eq. (3).